Primitives - Intégrales

2.2.6 Calculer :

a) $\int \cos^2(x)\,dx$ c) $\int \cos^3(x)\,dx$

b) $\int \sin^2(x)\,dx$ d) $\int \sin^4(x)\,dx$

a) $$\int \cos^2(x)\,dx = \int \frac{1+\cos(2x)}{2}\,dx = \frac{1}{2}\int dx + \frac{1}{2}\int \cos(2x)\,dx$$

$$= \frac{1}{2}\int dx + \frac{1}{2}\cdot\frac{1}{2}\int \underbrace{\cos(2x)}_{\cos(u)}\cdot\underbrace{2}_{u'}\,dx = \boxed{\frac{x}{2} + \frac{\sin(2x)}{4} + C}$$

b) $$\int \sin^2(x)\,dx = \int \left(1-\cos^2(x)\right)dx = \int dx - \int \underbrace{\cos^2(x)\,dx}_{\text{(question a)}}$$

$$= x - \left(\frac{x}{2} + \frac{\sin(2x)}{4}\right) + C$$

$$= x - \frac{x}{2} - \frac{\sin(2x)}{4} + C = \boxed{\frac{x}{2} - \frac{\sin(2x)}{4} + C}$$

c) $$\int \cos^3(x)\,dx = \int (1-\sin^2(x))\cos(x)\,dx = \int \left(\cos(x) - \cos(x)\sin^2(x)\right)dx$$

$$= \int \cos(x)\,dx - \int \underbrace{\sin^2(x)}_{u^2}\,\underbrace{\cos(x)}_{u'}\,dx = \boxed{\sin(x) - \frac{\sin^3(x)}{3} + C}$$

d) $$\int \sin^4(x)\,dx = \int \left(\frac{1-\cos(2x)}{2}\right)^2 dx = \int \left(\frac{1-2\cos(2x)+\cos^2(2x)}{4}\right)dx$$

$$= \frac{1}{4}\int \left(1-2\cos(2x)\right)dx + \frac{1}{4}\int \frac{1+\cos(4x)}{2}\,dx$$

$$= \frac{1}{4}\int 1 - \underbrace{2}_{u'}\,\underbrace{\cos(2x)}_{\cos(u)}\,dx + \frac{1}{8}\int 1\,dx + \frac{1}{32}\int \underbrace{\cos(4x)}_{\cos(u)}\cdot\underbrace{4}_{u'}\,dx$$

$$= \frac{x}{4} - \frac{\sin(2x)}{4} + \frac{x}{8} + \frac{\sin(4x)}{32} + C = \boxed{\frac{12x - 8\sin(2x) + \sin(4x) + C}{32}}$$

2.2.7 Calculer :

a) $\int \sin^2(x)\cos^2(x)\,dx$

b) $\int \sqrt{\sin(x)}\cos^3(x)\,dx$

c) $\int \sin(5x)\cos(3x)\,dx$

d) $\int \frac{\cos(x)}{2-\sin(x)}\,dx$

a) $$\int \sin^2(x)\cos^2(x)\,dx = \int \left(\frac{1-\cos(2x)}{2} \cdot \frac{1+\cos(2x)}{2}\right)dx$$

$$= \int \frac{1-\cos^2(2x)}{4}\,dx = \int \frac{1}{4}\,dx - \int \frac{\cos^2(2x)}{4}\,dx$$

$$= \frac{1}{4}\int dx - \frac{1}{4}\int \frac{1+\cos(4x)}{2}\,dx = \frac{1}{4}\int dx - \frac{1}{8}\int dx - \frac{1}{8}\int \cos(4x)\,dx$$

$$= \frac{1}{4}\int dx - \frac{1}{8}\int dx - \frac{1}{8}\cdot\frac{1}{4}\int \underbrace{\cos(4x)}_{\cos(u)}\cdot \underbrace{4}_{u'}\,dx$$

$$= \frac{1}{4}x - \frac{1}{8}x - \frac{1}{32}\sin(4x) + C$$

$$= \boxed{\frac{1}{8}x - \frac{1}{32}\sin(4x) + C}$$

b) $$\int \sqrt{\sin(x)}\cos^3(x)\,dx = \int \sqrt{\sin(x)}\cdot\cos(x)\cdot\cos^2(x)\,dx$$

$$= \int \sqrt{\sin(x)}\cdot\cos(x)\cdot\left(1-\sin^2(x)\right)dx = \int \sqrt{\sin(x)}\cos(x)\,dx - \int \sqrt{\sin(x)}\cos(x)\sin^2(x)\,dx$$

$$= \int \sin^{\frac{1}{2}}(x)\cos(x)\,dx - \int \sin^{\frac{1}{2}}(x)\cdot\sin^2(x)\cos(x)\,dx$$

$$= \int \underbrace{\sin^{\frac{1}{2}}(x)}_{u^{1/2}} \underbrace{\cos(x)}_{u'}\, dx - \int \underbrace{\sin^{\frac{5}{2}}(x)}_{u^{5/2}} \underbrace{\cos(x)}_{u'}\, dx$$

$$= \frac{2}{3} \sin^{\frac{3}{2}}(x) - \frac{2}{7} \sin^{\frac{7}{2}}(x) + C = \frac{2}{3} \sqrt{\sin^3(x)} - \frac{2}{7} \sqrt{\sin^7(x)} + C$$

$$= \boxed{\frac{2}{3} \sin(x) \sqrt{\sin(x)} - \frac{2}{7} \sin^3(x) \sqrt{\sin(x)} + C}$$

c) $\int \sin(5x) \cos(3x)\, dx$

Relations trigonométriques :

1) $\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$

$\Rightarrow \sin(5x + 3x) = \sin(5x)\cos(3x) + \cos(5x)\sin(3x) \quad (*)$

2) $\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$

$\Rightarrow \sin(5x - 3x) = \sin(5x)\cos(3x) - \cos(5x)\sin(3x) \quad (**)$

$\Rightarrow (*) + (**) \Rightarrow \sin(8x) + \sin(2x) = 2\sin(5x)\cos(3x)$

$$\Rightarrow \sin(5x)\cos(3x) = \frac{\sin(8x) + \sin(2x)}{2}$$

D'où $\int \sin(5x)\cos(3x)\, dx = \frac{1}{2} \int \left(\sin(8x) + \sin(2x)\right) dx$

$$= \frac{1}{2} \int \sin(8x)\, dx + \frac{1}{2} \int \sin(2x)\, dx$$

$$= \frac{1}{2} \cdot \frac{1}{8} \int \underbrace{\sin(8x)}_{\sin(u)} \cdot \underbrace{8}_{u'}\, dx + \frac{1}{2} \cdot \frac{1}{2} \int \underbrace{\sin(2x)}_{\sin(u)} \cdot \underbrace{2}_{u'}\, dx$$

$$= \boxed{-\frac{1}{16} \cos(8x) - \frac{1}{4} \cos(2x) + C}$$

d) $\displaystyle\int \frac{\cos(x)}{2-\sin(x)}\,dx = (-1)\int \frac{\overbrace{(-1)\cos(x)}^{u'}}{\underbrace{2-\sin(x)}_{u}}\,dx$

$= \boxed{-\ln\left(2-\sin(x)\right)+C}$

2.2.8 Calculer :

a) $\displaystyle\int (3x^2-2x+3)\,dx$

b) $\displaystyle\int \frac{3x^4-3x^2-7}{4x^2}\,dx$

c) $\displaystyle\int 7\sqrt[4]{x^3}\,dx$

d) $\displaystyle\int (\sqrt{x}-\sqrt[3]{x})\,dx$

e) $\displaystyle\int (2\sin(x)-3\cos(x))\,dx$

f) $\displaystyle\int \cos(2x)\,dx$

e) $\displaystyle\int \left(2\sin(x)-3\cos(x)\right)dx = 2\int \sin(x)\,dx - 3\int \cos(x)\,dx$

$= \boxed{-2\cos(x)-3\sin(x)+C}$

f) $\displaystyle\int \cos(2x)\,dx = \frac{1}{2}\int \underbrace{\cos(2x)}_{\cos(u)}\cdot\underbrace{2}_{u'}\,dx = \boxed{\frac{1}{2}\sin(2x)+C}$

g) $\displaystyle\int \left(\frac{5}{\cos^2(x)}+5\cos(x)\right)dx$

h) $\displaystyle\int \left(8\sin(x)+\frac{4}{\sqrt{2x}}\right)dx$

i) $\displaystyle\int (3x^2-7)^2\,dx$

j) $\displaystyle\int \sqrt{x}(x^2-5)\,dx$

k) $\displaystyle\int (3x-5)^6\,dx$

l) $\displaystyle\int \frac{12}{(4-3x)^4}\,dx$

m) $\displaystyle\int \sqrt[3]{(3x-8)^2}\,dx$

n) $\displaystyle\int \frac{6}{\cos^2(3x)}\,dx$

o) $\displaystyle\int x\sqrt{x^2+1}\,dx$

p) $\displaystyle\int \frac{2x-1}{\sqrt{x^2-x-1}}\,dx$

g) $\displaystyle\int \left(\frac{5}{\cos^2(x)}+5\cos(x)\right)dx = 5\int \frac{1}{\cos^2(x)}\,dx + 5\int \cos(x)\,dx$

$$= \boxed{5\underbrace{\tan(x)} + 5\sin(x) + C}$$

$$\left((\tan(x))' = 1 + \tan^2(x) = \frac{1}{\cos^2(x)} \right)$$

h) $$\int \left(8\sin(x) + \frac{4}{\sqrt{2x}} \right) dx = 8\int \sin(x)\,dx + \int \frac{4}{\sqrt{2x}}\,dx$$

$$= 8\int \sin(x)\,dx + \int 2 \cdot \frac{\overbrace{2}^{u'}}{\underbrace{\sqrt{2x}}_{\sqrt{u}}}\,dx = \boxed{-8\cos(x) + 4\sqrt{2x} + C}$$

k) $$\int (3x-5)^6\,dx = \frac{1}{3}\int \underbrace{(3x-5)^6}_{u^6}\ \underbrace{3}_{u'}\,dx = \frac{1}{3}\cdot\frac{1}{7}(3x-5)^7 + C$$

$$= \boxed{\frac{1}{21}(3x-5)^7 + C}$$

l) $$\int \frac{12}{(4-3x)^4}\,dx = \int 4 \cdot \frac{3}{(4-3x)^4}\,dx = 4(-1)\int (-1)\frac{3}{(4-3x)^4}\,dx$$

$$= -4\int \underbrace{(-3)}_{u'}\ \underbrace{(4-3x)^{-4}}_{u^{-4}}\,dx = -4 \cdot \frac{(4-3x)^{-3}}{(-3)} + C$$

$$= \frac{4}{3}(4-3x)^{-3} + C = \boxed{\frac{4}{3(4-3x)^3} + C}$$

m) $\displaystyle\int \sqrt[3]{(3x-8)^2}\,dx = \int (3x-8)^{2/3}\,dx = \frac{1}{3}\int \underbrace{(3x-8)^{2/3}}_{u^{2/3}}\,\underbrace{3}_{u'}\,dx$

$\displaystyle = \frac{1}{3}\,\frac{(3x-8)^{\frac{2}{3}+1}}{\frac{2}{3}+1} + C = \frac{1}{3}\,\frac{(3x-8)^{\frac{5}{3}}}{\frac{5}{3}} + C$

$\displaystyle = \frac{1}{3}\cdot\frac{3}{5}\,(3x-8)^{\frac{5}{3}} + C = \boxed{\frac{1}{5}\sqrt[3]{(3x-8)^5} + C}$

n) $\displaystyle\int \frac{6}{\cos^2(3x)}\,dx = \int 2\cdot\frac{3}{\cos^2(3x)}\,dx = 2\int \underbrace{\frac{1}{\cos^2(3x)}}_{(\tan(u))'}\cdot\underbrace{3}_{u'}\,dx$

$\displaystyle = \boxed{2\tan(3x) + C}$

o) $\displaystyle\int x\sqrt{x^2+1}\,dx = \int x\,(x^2+1)^{1/2}\,dx = \frac{1}{2}\int \underbrace{(x^2+1)^{1/2}}_{u^{1/2}}\,\underbrace{2x}_{u'}\,dx$

$\displaystyle = \frac{1}{2}\cdot\frac{2}{3}\,(x^2+1)^{\frac{3}{2}} + C = \boxed{\frac{1}{3}\sqrt{(x^2+1)^3} + C}$

p) $\displaystyle\int \frac{\overbrace{2x-1}^{u'}}{\underbrace{\sqrt{x^2-x-1}}_{\sqrt{u}}}\,dx = \boxed{2\sqrt{x^2-x-1} + C}$

2.2.9 Trouver l'expression mathématique de la fonction f, sachant que :

a) $f'(x) = 3x^2 - 4$, $f(5) = 54$;

b) $f''(x) = (x+1)(x-2)$, $f(1) = 8$, $f'(0) = 37/6$;

c) $f''(x) = \frac{1}{\sqrt{x}}$, $f'(9) = 2$, $f(1) = 2f(4)$.

a) * $f'(x) = 3x^2 - 4 \Rightarrow f(x) = \int (3x^2 - 4)\,dx = \int 3x^2\,dx - \int 4\,dx$

$\Rightarrow f(x) = 3\frac{x^3}{3} - 4x + C$

* $f(5) = 54 \Rightarrow f(5) = 3\cdot\frac{(5)^3}{3} - 4\cdot 5 + C = 54$

$\Rightarrow 125 - 20 + C = 54 \Rightarrow C = 54 - 125 + 20 = -51$

$\Rightarrow$ donc $\boxed{f(x) = x^3 - 4x - 51}$

b) * $f''(x) = (x+1)(x-2) \Rightarrow f'(x) = \int f''(x)\,dx = \int (x+1)(x-2)\,dx$

$\Rightarrow f'(x) = \int (x^2 - 2x + x - 2)\,dx = \int (x^2 - x - 2)\,dx = \frac{x^3}{3} - \frac{x^2}{2} - 2x + C$

* $f'(0) = \frac{37}{6} \Rightarrow \frac{0^3}{3} - \frac{0^2}{2} - 2\cdot 0 + C = \frac{37}{6} \Rightarrow C = \frac{37}{6}$

donc $f'(x) = \frac{x^3}{3} - \frac{x^2}{2} - 2x + \frac{37}{6}$

* $f(x) = \int f'(x)\,dx = \int \left(\frac{x^3}{3} - \frac{x^2}{2} - 2x + \frac{37}{6}\right)dx = \frac{x^4}{3\cdot 4} - \frac{x^3}{2\cdot 3} - \frac{2x^2}{2} + \frac{37}{6}x + C$

* $f(1) = 8 \Rightarrow \frac{1}{12} - \frac{1}{6} - \frac{2}{2} + \frac{37}{6} + C = 8$

$\Rightarrow \frac{1}{12} - \frac{1}{6} - 1 + \frac{37}{6} + C = 8 \Rightarrow C = 8 - \frac{1}{12} + \frac{1}{6} + 1 - \frac{37}{6}$

$$\Rightarrow C = \frac{96 - 1 + 2 + 12 - 74}{12} = \frac{35}{12}$$

d'où $\boxed{f(x) = \frac{x^4}{12} - \frac{x^3}{6} - x^2 + \frac{37x}{6} + \frac{35}{12}}$

c) * $f'(x) = \int \frac{1}{\sqrt{x}} \, dx = \int x^{-\frac{1}{2}} \, dx = \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C = 2x^{\frac{1}{2}} + C$

* $f'(9) = 2 \Rightarrow 2 \cdot \sqrt{9} + C = 2 \Rightarrow 2 \cdot 3 + C = 2$

$\Rightarrow C = -4 \Rightarrow f'(x) = 2\sqrt{x} - 4$

* $f(x) = \int (2\sqrt{x} - 4) \, dx = 2 \int (\sqrt{x} - 2) \, dx = 2 \cdot \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} - 4x + C$

$\Rightarrow f(x) = 2 \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - 4x + C = \frac{4}{3}\sqrt{x^3} - 4x + C$

* $f(1) = 2f(4)$

$\Rightarrow f(1) = \frac{4}{3}\sqrt{1} - 4 + C$ et $f(4) = \frac{4}{3}\sqrt{64} - 4 \cdot 4 + C$

$= \frac{32}{3} - 16 + C$

$\Rightarrow \frac{4}{3} - 4 + C = 2\left(\frac{32}{3} - 16 + C\right) = \frac{64}{3} - 32 + 2C$

$\Rightarrow -C = \frac{64}{3} - 32 - \frac{4}{3} + 4 = \frac{64 - 96 - 4 + 12}{3} = \frac{-24}{3} = -8$

$\Rightarrow C = 8$

d'où $\boxed{f(x) = \frac{4}{3}\sqrt{x^3} - 2x + 8}$