Primitives - Intégrales

2.2.6 Calculer :

a) $\int \cos^2(x)\,dx$ c) $\int \cos^3(x)\,dx$

b) $\int \sin^2(x)\,dx$ d) $\int \sin^4(x)\,dx$

a) $$\int \cos^2(x)\,dx = \int \frac{1+\cos(2x)}{2}\,dx = \frac{1}{2}\int dx + \frac{1}{2}\int \cos(2x)\,dx$$

$$= \frac{1}{2}\int dx + \frac{1}{2}\cdot\frac{1}{2}\int \underbrace{\cos(2x)}_{\cos(u)}\cdot\underbrace{2}_{u'}\,dx = \boxed{\frac{x}{2} + \frac{\sin(2x)}{4} + C}$$

b) $$\int \sin^2(x)\,dx = \int (1-\cos^2(x))\,dx = \int dx - \int \underbrace{\cos^2(x)\,dx}$$

$$= x - \left(\frac{x}{2} + \frac{\sin(2x)}{4}\right) + C \qquad \text{(question a)}$$

$$= x - \frac{x}{2} - \frac{\sin(2x)}{4} + C = \boxed{\frac{x}{2} - \frac{\sin(2x)}{4} + C}$$

c) $$\int \cos^3(x)\,dx = \int (1-\sin^2(x))\cos(x)\,dx = \int \left(\cos(x) - \cos(x)\sin^2(x)\right)dx$$

$$= \int \cos(x)\,dx - \int \underbrace{\sin^2(x)}_{u^2}\,\underbrace{\cos(x)}_{u'}\,dx = \boxed{\sin(x) - \frac{\sin^3(x)}{3} + C}$$

d) $$\int \sin^4(x)\,dx = \int \left(\frac{1-\cos^2(2x)}{2}\right)^2 dx = \int \left(\frac{1-2\cos(2x)+\cos^2(2x)}{4}\right)dx$$

$$= \frac{1}{4}\int (1-2\cos(2x))\,dx + \frac{1}{4}\int \frac{1+\cos(4x)}{2}\,dx$$

$$= \frac{1}{4}\int 1 - \underbrace{2}_{u'}\,\underbrace{\cos(2x)}_{\cos(u)}\,dx + \frac{1}{8}\int 1\,dx + \frac{1}{32}\int \underbrace{\cos(4x)}_{\cos(u)}\cdot\underbrace{4}_{u'}\,dx$$

$$= \frac{x}{4} - \frac{\sin(2x)}{4} + \frac{x}{8} + \frac{\sin(4x)}{32} + C = \boxed{\frac{12x - 8\sin(2x) + \sin(4x) + C}{32}}$$

2.2.7 Calculer :

a) $\displaystyle\int \sin^2(x)\cos^2(x)\,dx$

c) $\displaystyle\int \sin(5x)\cos(3x)\,dx$

b) $\displaystyle\int \sqrt{\sin(x)}\cos^3(x)\,dx$

d) $\displaystyle\int \frac{\cos(x)}{2-\sin(x)}\,dx$

a) $$\int \sin^2(x)\cos^2(x)\,dx = \int \left(\frac{1-\cos(2x)}{2}\cdot\frac{1+\cos(2x)}{2}\right)dx$$

$$= \int \frac{1-\cos^2(2x)}{4}\,dx = \int \frac{1}{4}\,dx - \int \frac{\cos^2(2x)}{4}\,dx$$

$$= \frac{1}{4}\int dx - \frac{1}{4}\int \frac{1+\cos(4x)}{2}\,dx = \frac{1}{4}\int dx - \frac{1}{8}\int dx - \frac{1}{8}\int \cos(4x)\,dx$$

$$= \frac{1}{4}\int dx - \frac{1}{8}\int dx - \frac{1}{8}\cdot\frac{1}{4}\int \underbrace{\cos(4x)}_{\cos(u)}\cdot\underbrace{4}_{u'}\,dx$$

$$= \frac{1}{4}x - \frac{1}{8}x - \frac{1}{32}\sin(4x) + C$$

$$= \boxed{\frac{1}{8}x - \frac{1}{32}\sin(4x) + C}$$

b) $$\int \sqrt{\sin(x)}\cos^3(x)\,dx = \int \sqrt{\sin(x)}\cdot\cos(x)\cdot\cos^2(x)\,dx$$

$$= \int \sqrt{\sin(x)}\cdot\cos(x)\cdot\left(1-\sin^2(x)\right)dx = \int \sqrt{\sin(x)}\cos(x)\,dx - \int \sqrt{\sin(x)}\cos(x)\sin^2(x)\,dx$$

$$= \int \sin^{\frac{1}{2}}(x)\cos(x)\,dx - \int \sin^{\frac{1}{2}}(x)\cdot\sin^2(x)\cos(x)\,dx$$

$$= \int \underbrace{\sin^{\frac{1}{2}}(x)}_{u^{1/2}} \underbrace{\cos(x)}_{u'}\,dx - \int \underbrace{\sin^{\frac{5}{2}}(x)}_{u^{5/2}} \underbrace{\cos(x)}_{u'}\,dx$$

$$= \frac{2}{3}\sin^{\frac{3}{2}}(x) - \frac{2}{7}\sin^{\frac{7}{2}}(x) + C = \frac{2}{3}\sqrt{\sin^3(x)} - \frac{2}{7}\sqrt{\sin^7(x)} + C$$

$$= \boxed{\frac{2}{3}\sin(x)\sqrt{\sin(x)} - \frac{2}{7}\sin^3(x)\sqrt{\sin(x)} + C}$$

c) $\int \sin(5x)\cos(3x)\,dx$

Relation trigonométriques :

1) $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$

$\Rightarrow \sin(5x+3x) = \sin(5x)\cos(3x) + \cos(5x)\sin(3x) \quad (*)$

2) $\sin(\alpha-\beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$

$\Rightarrow \sin(5x-3x) = \sin(5x)\cos(3x) - \cos(5x)\sin(3x) \quad (**)$

$\Rightarrow (*) + (**) \Rightarrow \sin(8x) + \sin(2x) = 2\sin(5x)\cos(3x)$

$\Rightarrow \sin(5x)\cos(3x) = \dfrac{\sin(8x)+\sin(2x)}{2}$

D'où $\int \sin(5x)\cos(3x)\,dx = \frac{1}{2}\int \left(\sin(8x) + \sin(2x)\right)dx$

$$= \frac{1}{2}\int \sin(8x)\,dx + \frac{1}{2}\int \sin(2x)\,dx$$

$$= \frac{1}{2}\cdot\frac{1}{8}\int \underbrace{\sin(8x)}_{\sin(u)}\cdot\underbrace{8}_{u'}\,dx + \frac{1}{2}\cdot\frac{1}{2}\int \underbrace{\sin(2x)}_{\sin(u)}\cdot\underbrace{2}_{u'}\,dx$$

$$= \boxed{-\frac{1}{16}\cos(8x) - \frac{1}{4}\cos(2x) + C}$$

d) $$\int \frac{\cos(x)}{2-\sin(x)}\,dx = (-1)\int \frac{\overbrace{(-1)\cos(x)}^{u'}}{\underbrace{2-\sin(x)}_{u}}\,dx$$

$$= \boxed{-\ln\big(2-\sin(x)\big) + C}$$

2.2.8 Calculer :

a) $\displaystyle\int (3x^2 - 2x + 3)\,dx$

b) $\displaystyle\int \frac{3x^4 - 3x^2 - 7}{4x^2}\,dx$

c) $\displaystyle\int 7\sqrt[4]{x^3}\,dx$

d) $\displaystyle\int (\sqrt{x} - \sqrt[3]{x})\,dx$

e) $\displaystyle\int (2\sin(x) - 3\cos(x))\,dx$

f) $\displaystyle\int \cos(2x)\,dx$

e) $$\int \big(2\sin(x) - 3\cos(x)\big)\,dx = 2\int \sin(x)\,dx - 3\int \cos(x)\,dx$$

$$= \boxed{-2\cos(x) - 3\sin(x) + C}$$

f) $$\int \cos(2x)\,dx = \frac{1}{2}\int \underbrace{\cos(2x)}_{\cos(u)} \cdot \underbrace{2}_{u'}\,dx = \boxed{\frac{1}{2}\sin(2x) + C}$$

g) $\displaystyle\int \left(\frac{5}{\cos^2(x)} + 5\cos(x)\right)dx$

h) $\displaystyle\int \left(8\sin(x) + \frac{4}{\sqrt{2x}}\right)dx$

i) $\displaystyle\int (3x^2 - 7)^2\,dx$

j) $\displaystyle\int \sqrt{x}(x^2 - 5)\,dx$

k) $\displaystyle\int (3x - 5)^6\,dx$

l) $\displaystyle\int \frac{12}{(4-3x)^4}\,dx$

m) $\displaystyle\int \sqrt[3]{(3x-8)^2}\,dx$

n) $\displaystyle\int \frac{6}{\cos^2(3x)}\,dx$

o) $\displaystyle\int x\sqrt{x^2+1}\,dx$

p) $\displaystyle\int \frac{2x-1}{\sqrt{x^2-x-1}}\,dx$

g) $$\int \left(\frac{5}{\cos^2(x)} + 5\cos(x)\right)dx = 5\int \frac{1}{\cos^2(x)}\,dx + 5\int \cos(x)\,dx$$

$$= \boxed{5\underbrace{\tan(x)}_{} + 5\sin(x) + C}$$

$$\left((\tan(x))' = 1 + \tan^2(x) = \frac{1}{\cos^2(x)} \right)$$

h) $\displaystyle\int \left(8\sin(x) + \frac{4}{\sqrt{2x}} \right) dx = 8\int \sin(x)\,dx + \int \frac{4}{\sqrt{2x}}\,dx$

$$= 8\int \sin(x)\,dx + \int 2 \cdot \frac{\overbrace{2}^{u'}}{\underbrace{\sqrt{2x}}_{\sqrt{u}}}\,dx = \boxed{-8\cos(x) + 4\sqrt{2x} + C}$$

k) $\displaystyle\int (3x-5)^6\,dx = \frac{1}{3}\int \underbrace{(3x-5)^6}_{u^6}\,\underbrace{3}_{u'}\,dx = \frac{1}{3}\cdot\frac{1}{7}(3x-5)^7 + C$

$$= \boxed{\frac{1}{21}(3x-5)^7 + C}$$

l) $\displaystyle\int \frac{12}{(4-3x)^4}\,dx = \int 4 \cdot \frac{3}{(4-3x)^4}\,dx = 4(-1)\int (-1)\frac{3}{(4-3x)^4}\,dx$

$$= -4\int \underbrace{(-3)}_{u'}\,\underbrace{(4-3x)^{-4}}_{u^{-4}}\,dx = -4\cdot\frac{(4-3x)^{-3}}{(-3)} + C$$

$$= \frac{4}{3}(4-3x)^{-3} + C = \boxed{\frac{4}{3(4-3x)^3} + C}$$

m) $\int \sqrt[3]{(3x-8)^2}\,dx = \int (3x-8)^{2/3}\,dx = \frac{1}{3}\int \underbrace{(3x-8)^{2/3}}_{u^{2/3}} \underbrace{3}_{u'}\,dx$

$= \frac{1}{3}\,\frac{(3x-8)^{\frac{2}{3}+1}}{\frac{2}{3}+1} + C = \frac{1}{3}\,\frac{(3x-8)^{\frac{5}{3}}}{\frac{5}{3}} + C$

$= \frac{1}{3}\cdot\frac{3}{5}\,(3x-8)^{\frac{5}{3}} + C = \boxed{\frac{1}{5}\sqrt[3]{(3x-8)^5} + C}$

n) $\int \frac{6}{\cos^2(3x)}\,dx = \int 2\cdot\frac{3}{\cos^2(3x)}\,dx = 2\int \underbrace{\frac{1}{\cos^2(3x)}}_{(\tan(u))'}\cdot \underbrace{3}_{u'}\,dx$

$= \boxed{2\tan(3x) + C}$

o) $\int x\sqrt{x^2+1}\,dx = \int x\,(x^2+1)^{1/2}\,dx = \frac{1}{2}\int \underbrace{(x^2+1)^{1/2}}_{u^{1/2}} \underbrace{2x}_{u'}\,dx$

$= \frac{1}{2}\cdot\frac{2}{3}\,(x^2+1)^{\frac{3}{2}} + C = \boxed{\frac{1}{3}\sqrt{(x^2+1)^3} + C}$

p) $\int \frac{\overbrace{2x-1}^{u'}}{\underbrace{\sqrt{x^2-x-1}}_{\sqrt{u}}}\,dx = \boxed{2\sqrt{x^2-x-1} + C}$

2.2.9 Trouver l'expression mathématique de la fonction f, sachant que :

a) $f'(x) = 3x^2 - 4$, $f(5) = 54$;

b) $f''(x) = (x+1)(x-2)$, $f(1) = 8$, $f'(0) = 37/6$;

c) $f''(x) = \dfrac{1}{\sqrt{x}}$, $f'(9) = 2$, $f(1) = 2f(4)$.

a) * $f'(x) = 3x^2 - 4 \Rightarrow f(x) = \int (3x^2 - 4)\,dx = \int 3x^2\,dx - \int 4\,dx$

$\Rightarrow f(x) = 3\dfrac{x^3}{3} - 4x + C$

* $f(5) = 54 \Rightarrow f(5) = 3\cdot\dfrac{(5)^3}{3} - 4\cdot 5 + C = 54$

$\Rightarrow 125 - 20 + C = 54 \Rightarrow C = 54 - 125 + 20 = -51$

$\Rightarrow$ donc $\boxed{f(x) = x^3 - 4x - 51}$

b) * $f''(x) = (x+1)(x-2) \Rightarrow f'(x) = \int f''(x)\,dx = \int (x+1)(x-2)\,dx$

$\Rightarrow f'(x) = \int (x^2 - 2x + x - 2)\,dx = \int (x^2 - x - 2)\,dx = \dfrac{x^3}{3} - \dfrac{x^2}{2} - 2x + C$

* $f'(0) = \dfrac{37}{6} \Rightarrow \dfrac{0^3}{3} - \dfrac{0^2}{2} - 2\cdot 0 + C = \dfrac{37}{6} \Rightarrow C = \dfrac{37}{6}$

donc $f'(x) = \dfrac{x^3}{3} - \dfrac{x^2}{2} - 2x + \dfrac{37}{6}$

* $f(x) = \int f'(x)\,dx = \int \left(\dfrac{x^3}{3} - \dfrac{x^2}{2} - 2x + \dfrac{37}{6}\right)dx = \dfrac{x^4}{3\cdot 4} - \dfrac{x^3}{2\cdot 3} - 2\dfrac{x^2}{2} + \dfrac{37}{6}x + C$

* $f(1) = 8 \Rightarrow \dfrac{1}{12} - \dfrac{1}{6} - \dfrac{2}{2} + \dfrac{37}{6} + C = 8$

$\Rightarrow \dfrac{1}{12} - \dfrac{1}{6} - 1 + \dfrac{37}{6} + C = 8 \Rightarrow C = 8 - \dfrac{1}{12} + \dfrac{1}{6} + 1 - \dfrac{37}{6}$

$\Rightarrow C = \dfrac{96 - 1 + 2 + 12 - 74}{12} = \dfrac{35}{12}$

d'où $\boxed{f(x) = \dfrac{x^4}{12} - \dfrac{x^3}{6} - x^2 + \dfrac{37x}{6} + \dfrac{35}{12}}$

c) * $f'(x) = \displaystyle\int \frac{1}{\sqrt{x}}\, dx = \int x^{-\frac{1}{2}}\, dx = \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C = 2x^{\frac{1}{2}} + C$

* $f'(9) = 2 \Rightarrow 2\cdot\sqrt{9} + C = 2 \Rightarrow 2\cdot 3 + C = 2$

$\Rightarrow C = -4 \Rightarrow f'(x) = 2\sqrt{x} - 4$

* $f(x) = \displaystyle\int \left(2\sqrt{x} - 4\right) dx = 2\int\left(\sqrt{x} - 2\right) dx = 2\cdot\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} - 4x + C$

$\Rightarrow f(x) = 2\dfrac{x^{\frac{3}{2}}}{\frac{3}{2}} - 4x + C = \dfrac{4}{3}\sqrt{x^3} - 4x + C$

* $f(1) = 2f(4)$

$\Rightarrow f(1) = \dfrac{4}{3}\sqrt{1} - 4 + C$ et $f(4) = \dfrac{4}{3}\sqrt{64} - 4\cdot 4 + C$

$= \dfrac{32}{3} - 16 + C$

$\Rightarrow \dfrac{4}{3} - 4 + C = 2\left(\dfrac{32}{3} - 16 + C\right) = \dfrac{64}{3} - 32 + 2C$

$\Rightarrow -C = \dfrac{64}{3} - 32 - \dfrac{4}{3} + 4 = \dfrac{64 - 96 - 4 + 12}{3} = \dfrac{-24}{3} = -8$

$\Rightarrow C = 8$

d'où $\boxed{f(x) = \dfrac{4}{3}\sqrt{x^3} - 2x + 8}$

2.2.11 Déterminer la fonction f sachant qu'elle admet pour asymptote la droite

$$x - 2y + 8 = 0$$

et que

$$f''(x) = -\frac{8}{x^3}$$

$* \quad f'(x) = \int f''(x)\,dx = \int -\frac{8}{x^3}\,dx = -8\int x^{-3}\,dx = -8\cdot\frac{x^{-2}}{-2} + C$

$\Rightarrow f'(x) = \frac{4}{x^2} + C$

$* \quad f(x) = \int f'(x)\,dx = \int\left(\frac{4}{x^2} + C\right)dx = 4\int(x^{-2} + C)\,dx$

$= 4\frac{x^{-1}}{-1} + Cx + d = -\frac{4}{x} + cx + d$

$*$ Asymptote : $x - 2y + 8 = 0 \Rightarrow y = \frac{x}{2} + 4$

Rappel : Théorème général de l'asymptote oblique :

$m = \lim_{x\to+\infty}\frac{f(x)}{x}$ et $h = \lim_{x\to+\infty}\left(f(x) - mx\right)$

$\Rightarrow$ d'où :

$* \quad m = \lim_{x\to+\infty}\frac{-\frac{4}{x} + cx + d}{x} = \lim_{x\to+\infty} -\frac{4}{x} + c + \frac{d}{x} = c$

$\Rightarrow c = \frac{1}{2}$

$* \quad h = \lim_{x\to+\infty}\left(-\frac{4}{x} + \frac{1}{2}x + d\right) = d \Rightarrow d = 4$

Donc $f(x) = -\frac{4}{x} + \frac{1}{2}x + 4 \Rightarrow \boxed{f(x) = \frac{x^2 + 8x - 8}{2x}}$

2.2.12 Calculer :

a) $\displaystyle\int_1^4 (x^2 - 2x + 3)\,dx$

b) $\displaystyle\int_{-1}^1 (2x^3 + 3x^2 + 2x - 1)\,dx$

c) $\displaystyle\int_{-1}^1 \sqrt[3]{x+1}\,dx$

d) $\displaystyle\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin(x)\,dx$

e) $\displaystyle\int_0^{\frac{\pi}{4}} (1 + \tan^2(x))\,dx$

f) $\displaystyle\int_0^2 (1 + 2x)^3\,dx$

g) $\displaystyle\int_1^4 \frac{dx}{\sqrt{x}}$

h) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^2(x)\cos(x)\,dx$

a) $$\int_1^4 (x^2 - 2x + 3)\,dx = \left(\frac{x^3}{3} - \frac{2x}{2} + 3x\right)\Big|_1^4 = \frac{64}{3} - 4 - \frac{1}{3} - 2 = \boxed{15}$$

b) $$\int_{-1}^1 (2x^3 + 3x^2 + 2x - 1)\,dx = \left(\frac{2x^4}{4} + \frac{3x^3}{3} + \frac{2x^2}{2} - x\right)\Big|_{-1}^1 = \frac{1}{2} + 1 - \frac{1}{2} - 1 = \boxed{0}$$

c) $$\int_{-1}^1 \sqrt[3]{x+1}\,dx = \int_{-1}^1 (x+1)^{\frac{1}{3}}\,dx = \frac{(x+1)^{\frac{1}{3}+1}}{\frac{1}{3}+1}\Big|_{-1}^1$$

$$= \frac{3}{4}(x+1)^{\frac{4}{3}}\Big|_{-1}^1 = \frac{3}{4}\sqrt[3]{(x+1)^4}\Big|_{-1}^1 = \frac{3}{4}\sqrt[3]{2^4} - \frac{3}{4}\sqrt[3]{0}$$

$$= \frac{3}{4}\sqrt[3]{2^3 \cdot 2} = \frac{3 \cdot 2}{4}\sqrt[3]{2} = \boxed{\frac{3}{2}\sqrt[3]{2}}$$

d) $$\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin(x)\,dx = (-1)\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin(x)(-1)\,dx = -\cos(x)\Big|_{\frac{\pi}{4}}^{\frac{3\pi}{4}}$$

$$= -\cos\left(\frac{3\pi}{4}\right) - \left(-\cos\left(\frac{\pi}{4}\right)\right) = -\left(-\frac{\sqrt{2}}{2}\right) - \left(-\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \boxed{\sqrt{2}}$$

e) $$\int_0^{\pi/4} \underbrace{\left(1+\tan^2(x)\right)}_{\tan(x)} dx = \tan(x)\Big|_0^{\pi/4} = \tan\left(\frac{\pi}{4}\right) - \tan(0) =$$

$$= 1 - 0 = \boxed{1}$$

f) $$\int_0^2 (1+2x)^3\, dx \qquad \frac{1}{2}\int_0^2 \underbrace{(1+2x)^3}_{u^3}\, \underbrace{2}_{u'}\, dx = \frac{1}{2}\,\frac{(1+2x)^4}{4}\Big|_0^2$$

$$= \frac{1}{8}\,(1+2x)^4\Big|_0^2 = \frac{1}{8}\,(1+4)^4 - \frac{1}{8}\cdot 1 = \frac{1}{8}\cdot 625 - \frac{1}{8} = \boxed{78}$$

g) $$\int_1^4 \frac{dx}{\sqrt{x}} = \int_1^4 x^{-\frac{1}{2}}\, dx = \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\Bigg|_1^4 = 2\,x^{\frac{1}{2}}\Big|_1^4$$

$$= 2\sqrt{x}\,\Big|_1^4 = 2\sqrt{4} - 2\sqrt{1} = 2\cdot 2 - 2 = \boxed{2}$$

h) $$\int_0^{\pi/2} \underbrace{\sin^2(x)}_{u^2}\,\underbrace{\cos(x)}_{u'}\, dx = \int_0^{\pi/2} \sin^2(x)\,\cos(x)\, dx = \frac{\sin^3(x)}{3}\Bigg|_0^{\pi/2}$$

$$= \frac{1}{3}\sin^3\left(\frac{\pi}{2}\right) - \frac{1}{3}\sin^3(0) = \frac{1}{3}(1)^3 - \frac{1}{3}\cdot 0 = \boxed{\frac{1}{3}}$$

2.2.13 Sachant que $\int_0^1 f(x)\,dx = 3$, $\int_1^2 f(x)\,dx = 4$ et $\int_2^3 f(x)\,dx = -8$, calculer :

a) $\int_0^2 f(x)\,dx$

b) $\int_0^1 3f(x)\,dx$

c) $\int_0^3 8f(x)\,dx$

d) $\int_3^1 2f(x)\,dx$

a) $\int_0^2 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx = 3 + 4 = \boxed{7}$

b) $\int_0^1 3f(x)\,dx = 3\int_0^1 f(x)\,dx = 3\cdot 3 = \boxed{9}$

c) $\int_0^3 8f(x)\,dx = 8\int_0^3 f(x)\,dx = 8\left(\int_0^2 f(x)\,dx + \int_2^3 f(x)\,dx\right) = 8\left(7+(-8)\right) = \boxed{-8}$

d) $\int_3^1 2f(x)\,dx = -\int_1^3 2f(x)\,dx = -2\int_1^3 f(x)\,dx$

$= -2\left(\int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx\right) = -2\left(4+(-8)\right) = \boxed{8}$

2.2.14 Montrer que pour une fonction f continue sur $[-a;a]$, on a :

a) $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$ lorsque f est paire ;

b) $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$ lorsque f est impaire.

! valeur en t

a) $$\int_{-a}^{0} f(t)\,dt = -\int_{0}^{-a} f(t)\,dt = -\int_{0}^{a} \underbrace{f(-x)}_{f(x)}\,(-1)\,dx = \int_{0}^{a} f(x)\,dx$$

poser $t=-x$

$$\int_{-a}^{a} f(x)\,dx = \underbrace{\int_{-a}^{0} f(x)\,dx}_{=\int_0^a f(x)\,dx} + \int_{0}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx \quad \text{(cqfd)}$$

! valeur en t

b) $$\int_{-a}^{0} f(t)\,dt = -\int_{0}^{-a} f(t)\,dt = -\int_{0}^{a} \underbrace{f(-x)}_{=-f(x)}\,(-1)\,dx = -\int_{0}^{a} f(x)\,dx$$

poser $t=-x$

$$\Rightarrow \int_{-a}^{a} f(x) = \underbrace{\int_{-a}^{0} f(x)\,dx}_{=-\int_0^a f(x)\,dx} + \int_{0}^{a} f(x)\,dx = 0 \quad \text{(cqfd)}$$

2.2.15 Déterminer les réels k pour lesquels on a :

a) $\displaystyle\int_{-1}^{2} kx^2\,dx = \frac{2}{3}$

b) $\displaystyle\int_{4}^{k} (x^2-3x+7)\,dx = \frac{129}{2}$

c) $\displaystyle\int_{0}^{k/2} \cos(t)\,dt = \frac{1}{2}$

d) $\displaystyle\int_{k}^{0} \frac{2}{(x+1)^3}\,dx = -\int_{0}^{k} \frac{3}{(x+3)^2}\,dx$

a) $\displaystyle\int_{-1}^{2} kx^2\,dx = \frac{2}{3} \Rightarrow k\frac{x^3}{3}\Big|_{-1}^{2} = \frac{2}{3} \Rightarrow k\cdot\frac{2^3}{3} - k\frac{(-1)^3}{3} = \frac{2}{3}$

$\Rightarrow \frac{8}{3}k + \frac{k}{3} = \frac{2}{3} \Rightarrow 3k = \frac{2}{3} \Rightarrow \boxed{k = \frac{2}{9}}$

b) $\displaystyle\int_{4}^{k} (x^2-3x+7)\,dx = \frac{129}{2} \Rightarrow \left(\frac{x^3}{3} - \frac{3x^2}{2} + 7x\right)\Big|_{4}^{k} = \frac{129}{2}$

$\Rightarrow \frac{k^3}{3} - \frac{3k^2}{2} + 7k - \frac{64}{3} - 4 = \frac{129}{2} \Rightarrow 2k^3 - 9k^2 + 42k - 539 = 0$

	2	-9	42	-539
7		14	35	
	2	5	77	0

$\Rightarrow (k-7)(\underbrace{2k^2+5k+77}_{\Delta<0}) = 0 \Rightarrow \boxed{k = 7}$

c) $\displaystyle\int_{0}^{k/2} \cos(t)\,dt = \frac{1}{2} \Rightarrow \sin(t)\Big|_{0}^{\frac{k}{2}} = \frac{1}{2} \Rightarrow \sin\left(\frac{k}{2}\right) - 0 = \frac{1}{2}$

$\Rightarrow \begin{cases} \frac{k_1}{2} = \frac{\pi}{6} + \ell\cdot 2\pi \\ \frac{k_2}{2} = \pi - \frac{\pi}{6} + \ell 2\pi \end{cases} \Rightarrow \begin{cases} k_1 = \frac{\pi}{3} + \ell 4\pi \\ k_2 = \frac{5\pi}{3} + \ell 4\pi \end{cases}$

$\Rightarrow \boxed{k \in \left\{\frac{\pi}{3} + \ell 4\pi \;\middle|\; \ell \in \mathbb{Z}\right\} \cup \left\{\frac{5\pi}{3} + \ell 4\pi \;\middle|\; \ell \in \mathbb{Z}\right\}}$

d) $\int_k^0 \frac{2}{(x+1)^3}\,dx = -\int_0^k \frac{3}{(x+3)^2}\,dx$

$\Leftrightarrow \int_k^0 2(x+1)^{-3}\,dx = -\int_0^k 3(x+3)^{-2}\,dx \Leftrightarrow 2\,\frac{(x+1)^{-2}}{-2}\Big|_k^0 = -3\,\frac{(x+3)^{-1}}{-1}\Big|_k$

$\Leftrightarrow -\frac{1}{(x+1)^2}\Big|_k^0 = +\frac{3}{x+3}\Big|_0^k \Leftrightarrow -\frac{1}{1} + \frac{1}{(k+1)^2} = \frac{3}{k+3} - 1$

$\Leftrightarrow -1 + \frac{1}{(k+1)^2} = \frac{3}{k+3} - 1 \Leftrightarrow \frac{1}{(k+1)^2} = \frac{3}{k+3} \Leftrightarrow k+3 = 3(k+1)^2$

$\Rightarrow k+3 = 3(k^2+2k+1) = k+3 = 3k^2+6k+3$

$\Rightarrow 3k^2+k=0 \Leftrightarrow k(3k+5)=0$

$\Rightarrow \boxed{k \in \left\{-\frac{5}{3} \,;\, 0\right\}}$

2.2.16 Déterminer la nature des extremums des fonctions suivantes :

a) $f : x \longmapsto \displaystyle\int_0^x (t^3 - t)\,dt$

b) $f : x \longmapsto \displaystyle\int_0^x \sqrt{t+1}\,dt$

2.2.17 Calculer les intégrales suivantes à l'aide du changement de variable indiqué :

a) $\displaystyle\int_{-2}^{0} x\sqrt{x+2}\,dx, \quad x = t^2 - 2$

b) $\displaystyle\int_{0}^{3} \sqrt{9-x^2}\,dx, \quad x = 3\sin(t)$

c) $\displaystyle\int_{1}^{2} \frac{1}{x\sqrt{x-1}}\,dx, \quad x = u^2 + 1$

d) $\displaystyle\int_{a}^{2a} x^3\sqrt{x^2-a^2}\,dx, \quad \text{avec } a > 0, \quad x = \sqrt{a^2+t^2}$

a) $\displaystyle\int_{-2}^{0} x\sqrt{x+2}\,dx$, $x = t^2 - 2 \Rightarrow dx = 2t\,dt$

$t^2 = x+2 \Rightarrow t = \sqrt{x+2}$

si $x = -2 \Rightarrow t = 0$

$x = 0 \Rightarrow t = \sqrt{2}$

$$\Rightarrow \int_0^{\sqrt{2}} (t^2-2)\sqrt{t^2-2+2}\;2t\,dt = \int_0^{\sqrt{2}} (t^2-2)\,t\cdot 2t\,dt$$

$$= \int_0^{\sqrt{2}} (t^2-2)\,2t^2\,dt = \int_0^{\sqrt{2}} (2t^4 - 4t^2)\,dt = \left[\frac{2t^5}{5} - \frac{4t^3}{3}\right]_0^{\sqrt{2}}$$

$$= \frac{2(\sqrt{2})^5}{5} - \frac{4(\sqrt{2})^3}{3} = \frac{2\cdot 4\cdot\sqrt{2}}{5} - \frac{4\cdot 2\cdot\sqrt{2}}{3} = \frac{8\sqrt{2}}{5} - \frac{8\sqrt{2}}{3}$$

$$= \frac{24\sqrt{2} - 40\sqrt{2}}{15} = \boxed{\frac{-16\sqrt{2}}{15}}$$

b) $\displaystyle\int_0^3 \sqrt{9-x^2}\,dx$, $x = 3\sin(t) \Rightarrow dx = 3\cos(t)\,dt$

$x = 0 \Rightarrow \sin(t) = 0 \Rightarrow t = 0$

$x = 3 \Rightarrow \sin(t) = 1 \Rightarrow t = \frac{\pi}{2}$

$$\Rightarrow \int_0^3 \sqrt{9-x^2}\,dx = \int_0^{\pi/2} \sqrt{9 - 9\sin^2(t)}\; 3\cos(t)\,dt$$

$$= \int_0^{\pi/2} \sqrt{9(1-\sin^2(t))}\; 3\cos(t)\,dt = \int_0^{\pi/2} 3\cos(t)\cdot 3\cos(t)\,dt$$

$$= \int_0^{\pi/2} 9\cos^2(t)\,dt = 9\int_0^{\pi/2} \frac{1+\cos(2t)}{2}\,dt$$

$$= \frac{9}{2}\int_0^{\pi/2} dt + \frac{9}{2}\int_0^{\pi/2} \cos(2t)\,dt = \frac{9}{2}\int_0^{\pi/2} dt + \frac{9}{2}\cdot\frac{1}{2}\int_0^{\pi/2} \cos(2t)\cdot 2\,dt$$

$$= \left[\frac{9}{2}t + \frac{9}{4}\sin(2t)\right]_0^{\pi/2} = \frac{9}{2}\cdot\frac{\pi}{2} + \frac{9}{4}\sin\left(2\cdot\frac{\pi}{2}\right) - \frac{9}{4}\sin(0) = \boxed{\frac{9\pi}{4}}$$

c) $\displaystyle\int_{+1}^2 \frac{1}{x\sqrt{x-1}}\,dx$, $x = u^2 + 1 \Rightarrow dx = 2u\,du$

$\Rightarrow u^2 = x - 1$

$x = +1 \Rightarrow u = 0$

$x = 2 \Rightarrow u = 1$

$$\Rightarrow \int_1^2 \frac{1}{x\sqrt{x-1}}\,dx = \int_0^1 \frac{1}{(u^2+1)\sqrt{u^2}}\,2u\,du$$

$$= \int_0^1 \frac{1}{u(u^2+1)} \, 2u \, du = \int_0^1 \frac{2}{u^2+1} \, du$$

On pose $u = \tan(t) \Rightarrow du = (1 + \tan^2(t)) \, dt$

$u = 0 \Rightarrow t = 0$

$u = 1 \Rightarrow t = \frac{\pi}{4}$

$$\Rightarrow \int_0^{\pi/4} \frac{2}{\cancel{\tan^2(t)+1}} \cancel{(1+\tan^2(t))} \, dt = \int_0^{\pi/4} 2 \, dt = 2t \Big|_0^{\pi/4}$$

$$= 2 \cdot \frac{\pi}{4} - 0 = \boxed{\frac{\pi}{2}}$$

d) $\displaystyle\int_a^{2a} x^3 \sqrt{x^2 - a^2} \, dx$, avec $a > 0$, $x = \sqrt{a^2 + t^2}$

$$\Rightarrow dx = \frac{1}{2} (a^2 + t^2)^{-\frac{1}{2}} \cdot 2t \, dt$$

$$= \frac{t}{\sqrt{a^2 + t^2}} \, dt$$

$x = \sqrt{a^2 + t^2} \Rightarrow x^2 = a^2 + t^2 \Rightarrow t^2 = x^2 - a^2$

$t = \sqrt{x^2 - a^2}$

$x = a \Rightarrow t = \sqrt{a^2 - a^2} = 0$

$x = 2a \Rightarrow t = \sqrt{4a^2 - a^2} = \sqrt{3a^2} = a\sqrt{3}$

$$\Rightarrow \int_a^{2a} x^3 \sqrt{x^2 - a^2} \, dx = \int_0^{a\sqrt{3}} \left(\sqrt{a^2+t^2}\right)^3 \cdot \sqrt{a^2 + t^2 - a^2} \, \frac{t \, dt}{\sqrt{a^2+t^2}}$$

$$= \int_0^{a\sqrt{3}} \frac{(a^2+t^2)^{3/2}\, t \cdot t}{(a^2+t^2)^{1/2}}\, dt = \int_0^{a\sqrt{3}} (a^2+t^2)^{\frac{3}{2}-\frac{1}{2}}\, t^2\, dt$$

$$= \int_0^{a\sqrt{3}} (a^2+t^2)\, t^2\, dt = \int_0^{a\sqrt{3}} (a^2 t^2 + t^4)\, dt$$

$$= a^2 \int_0^{a\sqrt{3}} t^2\, dt + \int_0^{a\sqrt{3}} t^4\, dt = \left[a^2 \frac{t^3}{3} + \frac{t^5}{5} \right]_0^{a\sqrt{3}}$$

$$= \frac{a^2 (a\sqrt{3})^3}{3} - 0 + \frac{(a\sqrt{3})^5}{5} - 0 = \frac{a^5\, 3\sqrt{3}}{3} + \frac{a^5 \cdot 9 \cdot \sqrt{3}}{5}$$

$$= a^5\sqrt{3} + \frac{9a^5\sqrt{3}}{5} = \frac{5a^5\sqrt{3} + 9a^5\sqrt{3}}{5} = \boxed{\frac{14a^5\sqrt{3}}{5}}$$

2.2.18 Calculer les intégrales suivantes en effectuant une intégration par parties :

a) $\int_0^{\pi} x\sin(x)\,dx$

b) $\int_0^{\pi/2} \sin(x)\cos(x)\,dx$

c) $\int_0^{\pi/4} \cos^2(x)\,dx$

d) $\int_0^{3} x\sqrt{1+x}\,dx$

Rappel :

$$\boxed{\int_a^b f(x)\cdot g'(x)\,dx = \Big[f(x)\cdot g(x)\Big]_a^b - \int_a^b f'(x)\cdot g(x)\,dx}$$

a) $\int_0^{\pi} x\sin(x)\,dx$

$\Rightarrow \quad f(x) = x \quad \Rightarrow \quad f'(x) = 1$

$g'(x) = \sin(x) \quad \Rightarrow \quad g(x) = -\cos(x)$

$$\Rightarrow \int_0^{\pi} x\sin(x)\,dx = -x\cdot\cos(x)\Big|_0^{\pi} - \int_0^{\pi} 1\cdot(-\cos(x))\,dx$$

$$= -x\cos(x)\Big|_0^{\pi} + \int_0^{\pi} \cos(x)\,dx$$

$$= -x\cos(x)\Big|_0^{\pi} + \sin(x)\Big|_0^{\pi} = \pi - 0 + 0 - 0 = \boxed{\pi}$$

b) $\int_0^{\pi/2} \sin(x)\cos(x)\,dx$

$\Rightarrow \quad f(x) = \sin(x) \quad \Rightarrow \quad f'(x) = \cos(x)$

$g'(x) = \cos(x) \quad \Rightarrow \quad g(x) = \sin(x)$

$$\Rightarrow \int_0^{\pi/2} \sin(x)\cos(x)\,dx = \sin(x)\cdot\sin(x)\Big|_0^{\pi/2} - \int_0^{\pi/2} \cos(x)\cdot\sin(x)\,dx$$

$$= \sin^2(x) \Big|_0^{\pi/2} - \int_0^{\pi/2} \sin(x) \cos(x)\, dx$$

On pose $\int_0^{\pi/2} \sin(x) \cos(x) = A$

donc : $A = \sin^2(x) \Big|_0^{\pi/2} - A \Rightarrow 2A = \sin^2(x) \Big|_0^{\pi/2}$

$$\Rightarrow A = \frac{1}{2} \sin^2(x) \Big|_0^{\pi/2} = \frac{1}{2} \left(\sin^2\left(\frac{\pi}{2}\right) - \sin^2(0) \right)$$

$$\Rightarrow A = \int_0^{\pi/2} \sin(x)\cos(x)\, dx = \frac{1}{2} - 0 = \boxed{\frac{1}{2}}$$

c) $\int_0^{\pi/4} \cos^2(x)\, dx = \int_0^{\pi/4} \cos(x) \cos(x)\, dx$

$\Rightarrow f(x) = \cos(x) \Rightarrow f'(x) = -\sin(x)$

$g'(x) = \cos(x) \Rightarrow g(x) = \sin(x)$

$$\Rightarrow \int_0^{\pi/4} \cos(x) \cos(x)\, dx = \cos(x) \sin(x) \Big|_0^{\pi/4} - \int_0^{\pi/4} \left(-\sin(x) \sin(x)\right) dx$$

$$= \cos(x) \sin(x) \Big|_0^{\pi/4} + \int_0^{\pi/4} \sin^2(x)\, dx$$

$$= \cos(x) \sin(x) \Big|_0^{\pi/4} + \int_0^{\pi/4} \left(1 - \cos^2(x)\right) dx$$

$$= \cos(x) \sin(x) \Big|_0^{\pi/4} + \int_0^{\pi/4} 1\, dx - \int_0^{\pi/4} \cos^2(x)\, dx$$

$$\underbrace{\int_0^{\pi/4} \cos^2(x)\,dx}_{A} = \cos(x)\sin(x)\Big|_0^{\pi/4} + x\Big|_0^{\pi/4} - \underbrace{\int_0^{\pi/4} \cos^2(x)\,dx}_{A}$$

$$2A = \cos(x)\sin(x)\Big|_0^{\pi/4} + x\Big|_0^{\pi/4} \Rightarrow A = \frac{1}{2}\cos(x)\sin(x)\Big|_0^{\pi/4} + \frac{1}{2}x\Big|_0^{\pi/4}$$

$$\Rightarrow A = \int_0^{\pi/4} \cos^2(x)\,dx = \frac{1}{2}\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2} - 0 + \frac{\pi}{8} - 0 = \boxed{\frac{2+\pi}{8}}$$

d) $\displaystyle\int_0^3 x\sqrt{1+x}\,dx$

$$\Rightarrow f(x) = x \Rightarrow f'(x) = 1$$

$$g'(x) = \sqrt{1+x} = (1+x)^{1/2} \Rightarrow g(x) = \frac{2}{3}(x+1)^{3/2}$$

$$\Rightarrow g(x) = \frac{2}{3}\sqrt{(x+1)^3}$$

$$\Rightarrow \int_0^3 x\sqrt{1+x}\,dx = x\cdot\frac{2}{3}\sqrt{(x+1)^3}\Big|_0^3 - \int_0^3 1\cdot\frac{2}{3}\sqrt{(x+1)^3}\,dx$$

$$= \frac{2x}{3}\sqrt{(x+1)^3}\Big|_0^3 - \frac{2}{3}\frac{(x+1)^{\frac{3}{2}+1}}{\frac{3}{2}+1}\Big|_0^3 = \frac{2x}{3}\sqrt{(x+1)^3}\Big|_0^3 - \frac{2}{3}\frac{(x+1)^{\frac{5}{2}}}{\frac{5}{2}}\Big|_0^3$$

$$= \frac{2x}{3}\sqrt{(x+1)^3}\Big|_0^3 - \frac{4}{15}\sqrt{(x+1)^5}\Big|_0^3 = 16 - 0 - \frac{128}{15} + \frac{4}{15} = \boxed{\frac{116}{15}}$$

2.2.19 Calculer les intégrales définies suivantes.

a) $\displaystyle\int_1^2 \frac{x}{x+6}\,dx$

b) $\displaystyle\int_0^4 \sqrt{x}(x+2)\,dx$

c) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^5(x)\cos(x)\,dx$

d) $\displaystyle\int_2^{\sqrt{20}} 3x\sqrt{x^2+5}\,dx$

e) $\displaystyle\int_0^3 \sqrt{9-x^2}\,dx$

f) $\displaystyle\int_2^3 \frac{5x-2}{x^2-x}\,dx$

g) $\displaystyle\int_{-1}^0 \frac{dx}{x^2+2x+2}$

h) $\displaystyle\int_0^4 x\sqrt{4-x}\,dx$ (ind : par parties)

a) $\displaystyle\int_1^2 \frac{x}{x+6}\,dx$ $\Rightarrow$ Division

$$\begin{array}{r|l} x & x+6 \\ \cline{2-2} -\;\underline{x+6} & 1 \\ 0\;\;-6 & \end{array}$$

$$\Rightarrow \frac{x}{x+6} = 1 - \frac{6}{x+6}$$

$$\Rightarrow \int_1^2 \left(1 - \frac{6}{x+6}\right) dx = \int_1^2 dx - 6\int_1^2 \frac{1}{x+6}\,dx$$

$$= x\Big|_1^2 - 6\ln(|x+6|)\Big|_1^2$$

$$= 2-1-6\left(\ln(8)-\ln(7)\right) = \boxed{1+6\ln\left(\frac{7}{8}\right)}$$

b) $$\int_0^4 \sqrt{x}\,(x+2)\,dx = \int_0^4 (x\sqrt{x}+2\sqrt{x})\,dx = \int_0^4 (x\cdot x^{\frac{1}{2}}+2x^{\frac{1}{2}})\,dx$$

$$= \int_0^4 \left(x^{\frac{3}{2}}+2x^{\frac{1}{2}}\right) dx = \int_0^4 x^{\frac{3}{2}}\,dx + 2\int_0^4 x^{1/2}\,dx$$

$$= \frac{x^{\frac{5}{2}}}{\frac{5}{2}}\Bigg|_0^4 + 2\,\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_0^4 = \frac{2}{5}x^{\frac{5}{2}}\Big|_0^4 + \frac{4}{3}x^{\frac{3}{2}}\Big|_0^4$$

$$= \frac{2}{5}\sqrt{x^5}\Big|_0^4 + \frac{4}{3}\sqrt{x^3}\Big|_0^4 = \frac{64}{5} + \frac{32}{3} - 0$$

$$= \boxed{\frac{352}{15}}$$

c) $$\int_0^{\pi/2} \underbrace{\sin^5(x)}_{u^5}\, \underbrace{\cos(x)}_{u'}\, dx = \frac{\sin^6(x)}{6}\Big|_0^{\pi/2} = \frac{1}{6} - 0 = \boxed{\frac{1}{6}}$$

d) $$\int_2^{\sqrt{20}} 3x\sqrt{x^2+5}\, dx = \frac{1}{2}\int_2^{\sqrt{20}} 3\, \underbrace{\sqrt{x^2+5}}_{u^{1/2}} \cdot \underbrace{2x}_{u'}\, dx$$

$$= \frac{3}{2}\, \frac{(x^2+5)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\Big|_2^{\sqrt{20}} = \frac{3}{2}\cdot\frac{2}{3}\,(x^2+5)^{\frac{3}{2}}\Big|_2^{\sqrt{20}}$$

$$= \sqrt{(x^2+5)^3}\Big|_2^{\sqrt{20}} = 125 - 27 = \boxed{98}$$

e) $$\int_0^3 \sqrt{9-x^2}\, dx$$

On pose $x = 3\sin(t) \Rightarrow dx = 3\cos(t)\, dt$

$x = 0 \Rightarrow \sin(t) = 0 \Rightarrow t = 0$

$x = 3 \Rightarrow 3\sin(t) = 3 \Rightarrow \sin(t) = 1 \Rightarrow t = \frac{\pi}{2}$

$$\Rightarrow \int_0^3 \sqrt{9-x^2}\, dx = \int_0^{\pi/2} \sqrt{9 - 9\sin^2(t)}\; 3\cos(t)\, dt$$

$$= \int_0^{\pi/2} \sqrt{9(1-\sin^2(t))}\; 3\cos(t)\, dt = \int_0^{\pi/2} 3\sqrt{\cos^2(t)}\; 3\cos(t)\, dt$$

$$= 9\int_0^{\pi/2} \cos^2(t)\,dt = 9\underbrace{\int_0^{\pi/2} \cos(t)\cos(t)\,dt}$$

intégration par parties → ex 2.2.18c

$$= 9\left(\frac{t}{2} + \frac{1}{2}\underbrace{\cos(t)\sin(t)}\right)\Bigg|_0^{\pi/2}$$

$\frac{1}{2}\sin(2t) = \cos(t)\sin(t)$

$$= 9\left(\frac{t}{2} + \frac{1}{4}\sin(2t)\right)\Bigg|_0^{\pi/2} = \frac{9t}{2}\Bigg|_0^{\pi/2} + \frac{9}{4}\sin(2t)\Bigg|_0^{\pi/2}$$

$$= \frac{9}{2}\cdot\frac{\pi}{2} - 0 + \frac{9}{4}\sin\left(2\cdot\frac{\pi}{2}\right) - \frac{9}{4}\sin(2\cdot 0)$$

$$= \frac{9\pi}{4} - 0 + 0 - 0 = \boxed{\frac{9\pi}{4}}$$

f) $\displaystyle\int_2^3 \frac{5x-2}{x^2-x}\,dx$

⇒ Décomposition en fractions simples :

$$\frac{5x-2}{x^2-x} = \frac{5x-2}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1} = \frac{A(x-1)+Bx}{x(x-1)}$$

$\Leftrightarrow\ 5x-2 = A(x-1)+Bx \Leftrightarrow 5x-2 = Ax - A + Bx$

$\Leftrightarrow\ 5x-2 = (A+B)x - A$

$$\Rightarrow \begin{cases} A+B = 5 \\ -A = -2 \end{cases} \Leftrightarrow \begin{cases} B = 3 \\ A = 2 \end{cases}$$

D'où $\displaystyle\frac{5x-2}{x^2-x} = \frac{2}{x} + \frac{3}{x-1}$

$$\Rightarrow \int_2^3 \frac{5x-2}{x^2-x}\,dx = \int_2^3 \left(\frac{2}{x} + \frac{3}{x-1}\right)dx = 2\int_2^3 \frac{1}{x}\,dx + 3\int_2^3 \frac{1}{x-1}\,dx$$

$$= 2\ln(|x|)\Big|_2^3 + 3\ln(|x-1|)\Big|_2^3$$

$$= 2\ln(3) - 2\ln(2) + 3\ln(2) - 3\ln(1)$$

$$= \ln(3)^2 - \ln(2)^2 + \ln(2)^3 - 0$$

$$= \ln(9) - \ln(4) + \ln(8) = \underbrace{\ln(9) + \ln(8)}_{\ln(72)} - \ln(4)$$

$$= \ln(72) - \ln(4) = \ln\left(\frac{72}{4}\right) = \boxed{\ln(18)}$$

9) $$\int_{-1}^{0} \frac{dx}{x^2+2x+2} = \int_{-1}^{0} \frac{dx}{x^2+2x+1+1} = \int_{-1}^{0} \frac{dx}{(x+1)^2+1}$$

On pose $x = \tan(t) - 1 \Rightarrow dx = (1+\tan^2(t))\,dt$

Si $x = -1 \Rightarrow \tan(t) = 0 \Rightarrow t = 0$

$x = 0 \Rightarrow \tan(t) = 1 \Rightarrow t = \frac{\pi}{4}$

$$\Rightarrow \int_{-1}^{0} \frac{dx}{(x+1)^2+1} = \int_0^{\pi/4} \frac{1+\tan^2(t)}{(\tan(t)-1+1)^2+1}\,dt = \int_0^{\pi/4} \frac{1+\tan^2(t)}{\tan^2 t+1}\,dt$$

$$= \int_0^{\pi/4} dt = t\Big|_0^{\pi/4} = \boxed{\frac{\pi}{4}}$$

h) $\int_0^4 x\sqrt{4-x}\,dx$

$\Rightarrow f(x) = x \Rightarrow f'(x) = 1$

$g'(x) = (4-x)^{1/2} \Rightarrow g(x) = -\dfrac{(4-x)^{\frac{1}{2}+1}}{\frac{1}{2}+1} = \dfrac{-(4-x)^{\frac{3}{2}}}{\frac{3}{2}} = -\dfrac{2}{3}\sqrt{(4-x)^3}$

$\Rightarrow \int_0^4 x\sqrt{4-x}\,dx = -x\cdot\dfrac{2}{3}\sqrt{(4-x)^3}\Big|_0^4 - \int_0^4 1\left(-\dfrac{2}{3}\right)\sqrt{(4-x)^3}\,dx$

$= -\dfrac{2x}{3}\sqrt{(4-x)^3}\Big|_0^4 + \dfrac{2}{3}\int_0^4 (4-x)^{3/2}\,dx = -\dfrac{2x}{3}\sqrt{(4-x)^3} + \dfrac{2}{3}(-1)\int_0^4 (4-x)^{3/2}(-1)\,dx$

$= \dfrac{2x}{3}\sqrt{(4-x)^3}\Big|_0^4 - \dfrac{2}{3}\,\dfrac{(4-x)^{\frac{3}{2}+1}}{\frac{3}{2}+1}\Big|_0^4 = \dfrac{2x}{3}\sqrt{(4-x)^3}\Big|_0^4 - \dfrac{2}{3}\cdot\dfrac{2}{5}\sqrt{(4-x)^5}\Big|_0^4$

$= \dfrac{2\cdot 4}{3}\sqrt{(4-4)^3} - \dfrac{2\cdot 0}{3}\sqrt{(4-0)^3} - \left(\dfrac{4}{15}\sqrt{(4-4)^0} - \dfrac{4}{15}\sqrt{(4-0)^5}\right)$

$= \dfrac{8}{3}\cdot 0 - 0 - \dfrac{4}{15}\cdot 0 + \dfrac{4}{15}\sqrt{1024} = \dfrac{32\cdot 4}{15} = \boxed{\dfrac{128}{15}}$

2.2.20 Calculer les intégrales suivantes en utilisant des formules trigonométriques.

a) $\displaystyle\int \sin^3(x)\,\cos^2(x)\,dx$

b) $\displaystyle\int \sqrt{\sin(x)}\,\cos^5(x)\,dx$

c) $\displaystyle\int \sin^2(3x)\,\cos^2(3x)\,dx$

d) $\displaystyle\int \sin(x+1)\,\cos(x-1)\,dx$

a) $\displaystyle\int \sin^3(x)\cos^2(x)\,dx = \int \sin(x)\cdot\sin^2(x)\cos^2(x)\,dx = \int \sin(x)\left(1-\cos^2(x)\right)\cos^2(x)\,dx$

$\displaystyle = \int \sin(x)\cos^2(x)\left(1-\cos^2(x)\right)dx = (-1)\int \underbrace{(-\sin(x))}_{u'}\underbrace{(\cos^2(x))}_{u^2}\,dx - (-1)\int \underbrace{(-\sin(x))}_{u'}\underbrace{(\cos^4(x))}_{u^4}\,dx$

$\displaystyle = \boxed{-\frac{\cos^3(x)}{3} + \frac{\cos^5(x)}{5} + C}$

b) $\displaystyle\int \sqrt{\sin(x)}\cdot\cos^5(x)\,dx = \int \sqrt{\sin(x)}\left(\cos^2(x)\right)^2\cos(x)\,dx$

$\displaystyle = \int \sqrt{\sin(x)}\cdot\left(1-\sin^2(x)\right)^2\cos(x)\,dx = \int \sin^{1/2}(x)\cos(x)\left(1+\sin^4(x)-2\sin^2(x)\right)dx$

$\displaystyle = \int \sin^{1/2}(x)\cos(x)\,dx + \int \sin^{1/2}(x)\cos(x)\cdot\sin^4(x)\,dx - \int \sin^{1/2}(x)\cos(x)\cdot 2\sin^2(x)\,dx$

$\displaystyle = \int \underbrace{\sin^{1/2}(x)}_{u^{1/2}}\underbrace{\cos(x)}_{u'}\,dx + \int \underbrace{\sin^{9/2}(x)}_{u^{9/2}}\underbrace{\cos(x)}_{u'}\,dx - 2\int \underbrace{\sin^{5/2}(x)}_{u^{5/2}}\underbrace{\cos(x)}_{u'}\,dx$

$\displaystyle = \frac{\sin^{\frac{3}{2}}(x)}{\frac{3}{2}} + \frac{\sin^{\frac{11}{2}}(x)}{\frac{11}{2}} - 2\,\frac{\sin^{\frac{7}{2}}(x)}{\frac{7}{2}} + C$

$\displaystyle = \boxed{\frac{2}{3}\sqrt{\sin^3(x)} + \frac{2}{11}\sqrt{\sin^{11}(x)} - \frac{4}{7}\sqrt{\sin^7(x)} + C}$

c) $\int \sin^2(3x)\cos^2(3x)\,dx = \int \big(\underbrace{\sin(3x)\cos(3x)}\big)^2\,dx$

Rappel : $\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$

$$= \int \left(\frac{\sin(6x)}{2}\right)^2 dx = \frac{1}{4}\int \sin^2(6x)\,dx$$

Rappel : $\cos(2\alpha) = 1 - 2\sin^2(\alpha)$

$$= \frac{1}{4}\int \frac{1-\cos(12x)}{2}\,dx = \frac{1}{8}\int dx - \frac{1}{8}\int \cos(12x)\,dx$$

$$= \frac{1}{8}x - \frac{1}{8}\cdot\frac{1}{12}\int \cos(12x)\cdot 12\,dx = \frac{x}{8} - \frac{1}{96}\sin(12x) + C$$

$$= \boxed{\frac{x}{8} - \frac{\sin(12x)}{96} + C}$$

d) $\int \sin(x+1)\cos(x-1)\,dx$

Rappel : $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$

$+ \quad \sin(\alpha-\beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$

$\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin(\alpha)\cos(\beta) \Rightarrow \sin(\alpha)\cos(\beta) = \frac{1}{2}\big(\sin(\alpha+\beta) + \sin(\alpha-\beta)\big)$

$$\Rightarrow \int \sin(x+1)\cos(x-1)\,dx = \int \frac{\sin(x+1+x-1) + \sin(x+1-x+1)}{2}\,dx$$

$$= \frac{1}{2}\int \big(\sin(2x) + \sin(2)\big)\,dx = \frac{1}{2}\underbrace{\int \sin(2x)\,dx}_{\frac{1}{2}\int \sin(2x)\cdot 2\,dx} + \frac{1}{2}\int \sin(2)\,dx$$

$$= \frac{1}{2}\cdot\frac{1}{2}\big(-\cos(2x)\big) + \frac{1}{2}x\sin(2) + C$$

$$= \boxed{-\frac{1}{4}\cos(2x) + \frac{1}{2}x\sin(2) + C}$$

e) $\displaystyle\int \sin(3x)\,\sin(4x)\,dx$

f) $\displaystyle\int \sin^2(x)\,\cos(3x)\,dx$

e) $\displaystyle\int \sin(3x)\sin(4x)\,dx$

Rappel :

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$-\quad \cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$

$$\cos(\alpha+\beta) - \cos(\alpha-\beta) = -2\sin(\alpha)\sin(\beta)$$

$$\Rightarrow \sin(\alpha)\sin(\beta) = -\frac{\cos(\alpha+\beta)-\cos(\alpha-\beta)}{2}$$

$$\Rightarrow \int \sin(3x)\sin(4x)\,dx = \int -\frac{1}{2}\Big(\cos(3x+4x)-\cos(3x-4x)\Big)\,dx$$

$$= -\frac{1}{2}\int \Big(\cos(7x) - \underbrace{\cos(-x)}_{\cos(x)}\Big)\,dx = -\frac{1}{2}\int\Big(\cos(7x)-\cos(x)\Big)\,dx$$

$$= -\frac{1}{2}\int \cos(7x)\,dx - \frac{1}{2}\int -\cos(x)\,dx = -\frac{1}{2}\int \cos(7x)\,dx + \frac{1}{2}\int \cos(x)\,dx$$

$$= -\frac{1}{2}\cdot\frac{1}{7}\sin(7x) + \frac{1}{2}\sin(x) + C = \boxed{-\frac{1}{14}\sin(7x) + \frac{1}{2}\sin(x) + C}$$

f) $\displaystyle\int \sin^2(x)\cos(3x)\,dx = \int \sin(x)\,\underbrace{\sin(x)\cos(3x)}_{\text{même principe que le d)}}\,dx$

$$\frac{1}{2}\Big(\sin(\alpha+\beta) + \sin(\alpha-\beta)\Big)$$

$$= \int \sin(x)\cdot\frac{1}{2}\Big(\sin(x+3x) + \sin(x-3x)\Big)\,dx = \frac{1}{2}\int \sin(x)\Big(\sin(4x)+\sin(-2x)\Big)\,dx$$

Rappel : $\sin(-\alpha) = -\sin(\alpha)$

$$= \frac{1}{2}\int \sin(x)\Big(\sin(4x)-\sin(2x)\Big)dx = \frac{1}{2}\int \underbrace{\sin(x)\sin(4x)}_{\text{même principe que le e)}}dx - \frac{1}{2}\int \underbrace{\sin(x)\sin(2x)}_{''}dx$$

$$= \frac{1}{4}\int\Big(\cos(3x)-\cos(5x)\Big)dx - \frac{1}{4}\int\big(\cos(x)-\cos(3x)\big)dx$$

$$= \frac{\sin(3x)}{12} - \frac{\sin(5x)}{20} - \frac{\sin(x)}{4} + \frac{\sin(3x)}{12} + C$$

$$= \boxed{\frac{\sin(3x)}{6} - \frac{\sin(x)}{4} - \frac{\sin(5x)}{20} + C}$$

2.2.21 Calculer

$$\int_1^3 \frac{x}{\sqrt{x+1}}\,dx$$

de trois manières différentes :

a) en effectuant le changement de variable $x = u - 1$;

b) en effectuant le changement de variable $x = t^2 - 1$;

c) en effectuant une intégration par parties.

a) $x = u-1 \Rightarrow dx = du$

$x = 1 \Rightarrow u = 2$

$x = 3 \Rightarrow u = 4$

$$\Rightarrow \int_1^3 \frac{x}{\sqrt{x+1}}dx = \int_2^4 \frac{u-1}{\sqrt{u-1+1}}du = \int_2^4 \frac{u-1}{\sqrt{u}}du = \int_2^4 (u-1)\,u^{-1/2}du$$

$$= \int_2^4\left(u\cdot u^{-\frac{1}{2}} - u^{-\frac{1}{2}}\right)du = \int_2^4\left(u^{\frac{1}{2}} - u^{-\frac{1}{2}}\right)du = \int_2^4 u^{\frac{1}{2}}du - \int_2^4 u^{-\frac{1}{2}}du$$

$$= \frac{2}{3} u^{\frac{3}{2}} \Big|_2^4 - 2u^{\frac{1}{2}} \Big|_2^4 = \frac{2}{3} \sqrt{u^3} \Big|_2^4 - 2\sqrt{u} \Big|_2^4$$

$$= \frac{2}{3} \sqrt{4^3} - \frac{2}{3} \sqrt{2^3} - 2\sqrt{4} + 2\sqrt{2} = \frac{2}{3} \cdot 8 - \frac{4}{3} \sqrt{2} - 4 + 2\sqrt{2}$$

$$= \frac{16}{3} - 4 - \frac{4}{3}\sqrt{2} + 2\sqrt{2} = \frac{16}{3} - \frac{12}{3} - \frac{4\sqrt{2}}{3} + \frac{6\sqrt{2}}{3} = \boxed{\frac{4 + 2\sqrt{2}}{3}}$$

b) $x = t^2 - 1 \Rightarrow dx = 2t\,dt$

$x = 1 \Rightarrow t^2 = 2 \Rightarrow t = \sqrt{2}$

$x = 3 \Rightarrow t^2 = 4 \Rightarrow t = 2$

$$\Rightarrow \int_{\sqrt{2}}^{2} \frac{t^2 - 1}{\sqrt{t^2 - 1 + 1}} 2t\,dt = \int_{\sqrt{2}}^{2} \frac{t^2 - 1}{t} 2t\,dt = \int_{\sqrt{2}}^{2} (2t^2 - 2)\,dt$$

$$= 2\int_{\sqrt{2}}^{2} t^2\,dt - 2\int_{\sqrt{2}}^{2} dt = 2\frac{t^3}{3}\Big|_{\sqrt{2}}^{2} - 2t\Big|_{\sqrt{2}}^{2} = 2 \cdot \frac{8}{3} - \frac{2(\sqrt{2})^3}{3} - \left(2 \cdot 2 - 2\sqrt{2}\right)$$

$$= \frac{16}{3} - \frac{4\sqrt{2}}{3} - 4 + 2\sqrt{2} = \frac{16 - 4\sqrt{2} - 12 + 6\sqrt{2}}{3} = \boxed{\frac{4 + 2\sqrt{2}}{3}}$$

c) Intégration par parties

$$\int_1^3 \frac{x}{\sqrt{x+1}}\,dx = \int_1^3 x\,(x+1)^{-\frac{1}{2}}\,dx$$

$f(x) = x \Rightarrow f'(x) = 1$

$g'(x) = (x+1)^{-\frac{1}{2}} \Rightarrow g(x) = \frac{(x+1)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = \frac{(x+1)^{\frac{1}{2}}}{\frac{1}{2}} = 2(x+1)^{\frac{1}{2}}$

$$= \int_1^3 \frac{x}{\sqrt{x+1}} = x \cdot 2(x+1)^{\frac{1}{2}} \Big|_1^3 - \int_1^3 1 \cdot 2(x+1)^{\frac{1}{2}}\,dx$$

$$= 2x\sqrt{x+1}\,\Big|_1^3 - 2\int_1^3 (x+1)^{\frac{1}{2}}\,dx = 2x\sqrt{x+1}\,\Big|_1^3 - 2\frac{(x+1)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\Big|_1^3$$

$$= 2x\sqrt{x+1}\,\Big|_1^3 - 2\frac{(x+1)^{\frac{3}{2}}}{\frac{3}{2}}\Big|_1^3 = 2x\sqrt{x+1}\,\Big|_1^3 - \frac{4}{3}\sqrt{(x+1)^3}\,\Big|_1^3$$

$$= 2\cdot 3\sqrt{3+1} - 2\cdot 1\sqrt{1+1} - \frac{4}{3}\sqrt{(3+1)^3} + \frac{4}{3}\sqrt{(1+1)^3} = 6\cdot 2 - 2\sqrt{2} - \frac{4}{3}\sqrt{64} + \frac{4}{3}\sqrt{(1+1)^3}$$

$$= 12 - 2\sqrt{2} - \frac{4}{3}\cdot 8 + \frac{4}{3}\cdot 2\sqrt{2} = 12 - 2\sqrt{2} - \frac{32}{3} + \frac{8}{3}\sqrt{2} = \frac{36 - 6\sqrt{2} - 32 + 8\sqrt{2}}{3}$$

$$= \boxed{\frac{4 + 2\sqrt{2}}{3}}$$

2.2.22 Calculer, si possible, les intégrales généralisées ci-dessous :

a) $\displaystyle\int_1^{+\infty} \frac{2}{x^2}\,dx$

b) $\displaystyle\int_{-\infty}^{-2} \frac{1}{(x+1)^3}\,dx$

c) $\displaystyle\int_3^{+\infty} \frac{5+y}{y^3}\,dy$

d) $\displaystyle\int_0^{+\infty} \frac{x^2+1}{x^2}\,dx$

e) $\displaystyle\int_0^{2} \frac{2}{x^2}\,dx$

f) $\displaystyle\int_0^{4} t^{-3/2}\,dt$

g) $\displaystyle\int_{-\infty}^{+\infty} \frac{3}{z^2+1}\,dz$

h) $\displaystyle\int_0^{\pi^2} \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx$

<u>rappel :</u>

$$\boxed{\begin{array}{l} * \displaystyle\int_a^{+\infty} f(x)\,dx = \lim_{b\to+\infty} \int_a^b f(x)\,dx \\[2ex] * \displaystyle\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx \\[2ex] * \displaystyle\int_{-\infty}^{+\infty} f(x)\,dx = \lim_{\substack{a\to-\infty \\ b\to+\infty}} \int_a^b f(x)\,dx \end{array}}$$

a) $$\int_1^{+\infty} \frac{2}{x^2}\,dx = \lim_{b\to+\infty} \int_1^b \frac{2}{x^2}\,dx = \lim_{b\to+\infty} \int_1^b 2x^{-2}\,dx$$

$$= \lim_{b\to+\infty} \left.\frac{2x^{-1}}{-1}\right|_1^b = \lim_{b\to+\infty} \left.-\frac{2}{x}\right|_1^b = \lim_{b\to+\infty}\left(-\frac{2}{b}+2\right) = \boxed{2}$$

ou $$\int_1^{+\infty} \frac{2}{x^2}\,dx = \left.-\frac{2}{x}\right|_1^{+\infty} = -\frac{2}{+\infty} + \frac{2}{1} = 2$$

b) $\int_{-\infty}^{-2} \frac{1}{(x+1)^3}\,dx = \int_{-\infty}^{-2} (x+1)^{-3}\,dx = \frac{(x+1)^{-2}}{-2}\Bigg|_{-\infty}^{-2}$

$= -\frac{1}{2(x+1)^2}\Bigg|_{-\infty}^{-2} = -\frac{1}{2(-2+1)^2} + \frac{1}{2(+\infty+1)^2} = \boxed{-\frac{1}{2}}$

$\downarrow$ 0 quand $x \to +\infty$

c) $\int_{3}^{+\infty} \frac{5+y}{y^3}\,dy = \int_{3}^{+\infty} (5+y)\,y^{-3}\,dy = \int_{3}^{+\infty} (5y^{-3} + y^{-2})\,dy$

$\Rightarrow \left(5\,\frac{y^{-2}}{-2} + \frac{y^{-1}}{-1}\right)\Bigg|_{3}^{+\infty} = \left(-\frac{5}{2y^2} - \frac{1}{y}\right)\Bigg|_{3}^{+\infty}$

$= \underbrace{\frac{-5}{2(+\infty)^2}}_{0} + \frac{5}{2\cdot 3^2} - \underbrace{\frac{1}{(+\infty)}}_{0} + \frac{1}{3} = \frac{5}{18} + \frac{1}{3} = \frac{5+6}{18} = \boxed{\frac{11}{18}}$

d) $\int_{0}^{+\infty} \frac{x^2+1}{x^2}\,dx = \int_{0}^{+\infty} \left(1 + \frac{1}{x^2}\right)dx = \int_{0}^{+\infty} (1 + x^{-2})\,dx$

$= \left(x + \frac{x^{-1}}{-1}\right)\Bigg|_{0}^{+\infty} = \left(x - \frac{1}{x}\right)\Bigg|_{0}^{+\infty} = +\infty - 0 - \frac{1}{+\infty} + 0$

$= \boxed{+\infty}$ $\Rightarrow$ L'intégrale généralisée diverge.

e) $\int_{0}^{2} \frac{2}{x^2}\,dx = \int_{0}^{2} 2x^{-2}\,dx = -2\,\frac{x^{-1}}{-1}\Bigg|_{0}^{2} = \frac{-2}{x}\Bigg|_{0}^{2} = -1 + \underbrace{\frac{2}{0}}_{+\infty} = \boxed{+\infty}$

$\Rightarrow$ L'intégrale généralisée diverge.

f) $\displaystyle\int_0^4 t^{-\frac{3}{2}}\,dt = \left.\frac{t^{-\frac{3}{2}+1}}{-\frac{3}{2}+1}\right|_0^4 = \left.\frac{t^{-\frac{1}{2}}}{-\frac{1}{2}}\right|_0^4 = \left.-\frac{2}{\sqrt{t}}\right|_0^4 = -\frac{2}{\sqrt{4}} + \underbrace{\frac{2}{\sqrt{0}}}_{+\infty} = \boxed{+\infty}$

$\Rightarrow$ L'intégrale généralisée diverge.

g) $\displaystyle\int_{-\infty}^{+\infty} \frac{3}{z^2+1}\,dz \Rightarrow$ poser $z = \tan(x)$

$\Rightarrow dz = (1+\tan^2(x))\,dx$

$\Rightarrow z = -\infty \Rightarrow x = -\frac{\pi}{2}$

$z = +\infty \Rightarrow x = +\frac{\pi}{2}$

$\displaystyle\Rightarrow \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{3}{\tan^2(x)+1}\,(1+\tan^2(x))\,dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 3\,dx = 3x\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$

$\displaystyle = 3\cdot\frac{\pi}{2} - 3\left(-\frac{\pi}{2}\right) + \frac{3\pi}{2} + \frac{3\pi}{2} = \frac{6\pi}{2} = \boxed{3\pi}$

h) $\displaystyle\int_0^{\pi^2} \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx \Rightarrow$ poser $x = t^2 \Rightarrow dx = 2t\,dt$

$\Rightarrow x = 0 \Rightarrow t = 0$

$x = \pi^2 \Rightarrow t = \sqrt{\pi^2} = \pi$

$\displaystyle\Rightarrow \int_0^{\pi} \frac{\sin(t)}{t}\,2t\,dt = \int_0^{\pi} 2\sin(t)\,dt = 2\underbrace{\int_0^{\pi}\sin(t)\,dt}_{(-\cos(t))}$

$\displaystyle = 2(-1)\cos(t)\Big|_0^{\pi} = -2\cos(t)\Big|_0^{\pi}$

$= -2\cos(\pi) + 2\cos(0) = 2+2 = \boxed{4}$