Corrigé

3MRJ

1.1.3 On donne $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. Déterminer toutes les matrices B carrées d'ordre 2 telles que $AB = BA$.

$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow AB = \begin{pmatrix} a-c & b-d \\ a+c & b+d \end{pmatrix}$; $BA = \begin{pmatrix} a+b & -a+b \\ c+d & -c+d \end{pmatrix}$

$\Rightarrow c = -b \quad a = d \Rightarrow \boxed{B = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}}$

1.1.5 Soit $A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ et $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Montrer que $(AB)(AB) = AB$.

$AB = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \Rightarrow \underline{(AB)(AB) = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = AB}$

1.1.6 Soit α, β, γ, δ des nombres réels, $A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$ et $B = \begin{pmatrix} \gamma & \delta \\ -\delta & \gamma \end{pmatrix}$. Montrer que $AB = BA$.

$\underline{AB = \begin{pmatrix} \alpha\gamma - \beta\delta & \alpha\delta + \beta\gamma \\ -\beta\gamma - \alpha\delta & -\beta\delta + \alpha\gamma \end{pmatrix} = BA}$

1.1.7 Soit $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Trouver une matrice B telle que $AB \neq 0$ et $BA = 0$.

$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow AB = \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix}$; $BA = \begin{pmatrix} b & 0 \\ d & 0 \end{pmatrix}$

$\Rightarrow a \neq 0 \quad b = 0 \quad d = 0 \Rightarrow \underline{B = \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}}$, $a \in \mathbb{R}^*$ et $c \in \mathbb{R}$

1.1.11 Résoudre les systèmes suivants :

a) $\begin{cases} 2x + y - z = 1 \\ x + 2y + z = 8 \\ 3x - y + 2z = 7 \end{cases}$ b) $\begin{cases} x - y + z = 0 \\ -x + y + z = 10 \\ x + y - z = -2 \end{cases}$ c) $\begin{cases} x + 2y - 5z + 4t = 1 \\ 2x - 3y + 2z + 3t = 18 \\ 4x - 7y + z - 6t = -5 \\ x + y - z + t = 1 \end{cases}$

a) $\begin{cases} 2x + y - z = 1 \\ x + 2y + z = 8 \\ 3x - y + 2z = 7 \end{cases}$

$$\Leftrightarrow \left(\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 1 & 2 & 1 & 8 \\ 3 & -1 & 2 & 7 \end{array}\right) \xrightarrow[L_3 = L_3 + 2L_1]{L_2 = L_2 + L_1} \left(\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 3 & 3 & 0 & 9 \\ 7 & 1 & 0 & 9 \end{array}\right) \xrightarrow[L_3 = L_3 - \frac{1}{3}L_2]{L_2 = \frac{1}{3}L_2}$$

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 1 & 1 & 0 & 3 \\ 6 & 0 & 0 & 6 \end{array}\right) \xrightarrow[L_1 \leftrightarrow L_3]{L_3 = \frac{1}{6}L_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 3 \\ 2 & 1 & -1 & 1 \end{array}\right) \xrightarrow[L_3 = L_3 - 2L_1]{L_2 = L_2 - L_1}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & -1 & -1 \end{array}\right) \xrightarrow{L_3 = L_2 - L_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\Rightarrow \boxed{S = \{(1;2;3)\}}$$

b) $\begin{cases} x - y + z = 0 \\ -x + y + z = 10 \\ x + y - z = -2 \end{cases}$

$$\Leftrightarrow \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 1 & 1 & 10 \\ 1 & 1 & -1 & -2 \end{array}\right) \xrightarrow[\substack{L_2 = L_2 + L_3 \\ L_3 = L_1 + L_2}]{L_1 = L_1 + L_3} \left(\begin{array}{ccc|c} 2 & 0 & 0 & -2 \\ 0 & 2 & 0 & 8 \\ 0 & 0 & 2 & 10 \end{array}\right)$$

$$\xrightarrow[\substack{L_2 = \frac{1}{2} L_2 \\ L_3 = \frac{1}{2} L_3}]{L_1 = \frac{1}{2} L_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{array}\right) \longrightarrow \boxed{S = \{(-1 ; 4 ; 5)\}}$$

c) $\begin{cases} x + 2y - 5z + 4t = 1 \\ 2x - 3y + 2z + 3t = 18 \\ 4x - 7y + z - 6t = -5 \\ x + y - z + t = 1 \end{cases}$

$$\left(\begin{array}{cccc|c} 1 & 2 & -5 & 4 & 1 \\ 2 & -3 & 2 & 3 & 18 \\ 4 & -7 & 1 & -6 & -5 \\ 1 & 1 & -1 & 1 & 1 \end{array}\right) \xrightarrow[\substack{L_3 = L_3 - 4L_1 \\ L_4 = L_1 - L_4}]{\substack{L_2 \leftrightarrow L_1 \\ L_2 = L_2 - 2L_1}} \left(\begin{array}{cccc|c} 2 & -3 & 2 & 3 & 18 \\ 0 & -7 & 12 & -5 & 16 \\ 0 & -15 & 21 & -22 & -9 \\ 0 & 1 & -4 & 3 & 0 \end{array}\right)$$

$$\xrightarrow[L_3 = L_3 + 15L_4]{L_2 = L_2 + 7L_4} \left(\begin{array}{cccc|c} 2 & -3 & 2 & 3 & 18 \\ 0 & 0 & -16 & 16 & 16 \\ 0 & 0 & -39 & 23 & -9 \\ 0 & 1 & -4 & 3 & 0 \end{array}\right) \xrightarrow{L_2 \leftrightarrow L_4} \left(\begin{array}{cccc|c} 2 & -3 & 2 & 3 & 18 \\ 0 & 1 & -4 & 3 & 0 \\ 0 & 0 & -39 & 23 & -9 \\ 0 & 0 & -16 & 16 & 16 \end{array}\right)$$

$$\xrightarrow{L_3 \leftrightarrow L_4} \left(\begin{array}{cccc|c} 2 & -3 & 2 & 3 & 18 \\ 0 & 1 & -4 & 3 & 0 \\ 0 & 0 & -16 & 16 & 16 \\ 0 & 0 & -39 & 23 & -9 \end{array}\right) \xrightarrow{L_3 = -\frac{1}{16} L_3} \left(\begin{array}{cccc|c} 2 & -3 & 2 & 3 & 18 \\ 0 & 1 & -4 & 3 & 0 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & -39 & 23 & -9 \end{array}\right)$$

$$\xrightarrow{L_4 = L_4 + 39L_3} \left(\begin{array}{cccc|c} 2 & -3 & 2 & 3 & 18 \\ 0 & 1 & -4 & 3 & 0 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & -16 & -48 \end{array}\right) \xrightarrow{L_4 = -\frac{1}{16} L_4} \left(\begin{array}{cccc|c} 2 & -3 & 2 & 3 & 18 \\ 0 & 1 & -4 & 3 & 0 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right)$$

$$\xrightarrow[L_3 = L_3 + L_4]{\substack{L_1 = L_1 - 3L_4 \\ L_2 = L_2 - 3L_4}} \left(\begin{array}{cccc|c} 2 & -3 & 2 & 0 & 9 \\ 0 & 1 & -4 & 0 & -9 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right) \xrightarrow[L_2 = L_2 + 4L_3]{L_1 = L_1 - 2L_3} \left(\begin{array}{cccc|c} 2 & -3 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right)$$

$$\xrightarrow{L_1 = L_1 + 3L_2} \left(\begin{array}{cccc|c} 2 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right) \xrightarrow{L_1 = \frac{1}{2} L_1} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right)$$

$$\Rightarrow \boxed{S = \{ (1 ; -1 ; 2 ; 3) \}}$$

1.1.12 Résoudre les systèmes suivants :

a) $\begin{cases} x+2y-3z=0 \\ 5x-3y+z=0 \end{cases}$ b) $\begin{cases} x+y+z=0 \\ -2x+5y+2z=1 \\ 8x+y+4z=-1 \end{cases}$ c) $\begin{cases} x+y+2z=4 \\ 3x-y+z=3 \\ -3x+5y+4z=5 \end{cases}$

d) $\begin{cases} 2x+y+3z=3 \\ 3x-y+4z=2 \\ 4x+y-z=5 \\ x+y+z=4 \end{cases}$ e) $\begin{cases} x-3y+z-t=0 \\ 2x+y-z+2t=0 \end{cases}$ f) $\begin{cases} x+5y+4z-13t=3 \\ 3x-y+2z+5t=2 \\ 2x+2y+3z-4t=1 \end{cases}$

a) $\begin{cases} x+2y-3z=0 \\ 5x-3y+z=0 \end{cases} \iff \left(\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 5 & -3 & 1 & 0 \end{array}\right) \xrightarrow[L_2=L_2-5L_1]{L_1=L_1+3L_2}$

$\left(\begin{array}{ccc|c} 16 & -7 & 0 & 0 \\ 0 & -13 & 16 & 0 \end{array}\right) \xrightarrow[L_2=\frac{1}{16}L_2]{L_1=\frac{1}{16}L_1} \left(\begin{array}{ccc|c} 1 & -7/16 & 0 & 0 \\ 0 & -13/16 & 1 & 0 \end{array}\right)$

$\iff \begin{cases} x - \frac{7}{16}y = 0 \\ -\frac{13}{16}y + z = 0 \end{cases}$ posons $\frac{y}{16} = \alpha \Rightarrow y = 16\alpha$

$\Rightarrow x = \frac{7}{16}y = 7\alpha$

$z = \frac{13}{16}y = 13\alpha$

donc $\boxed{S = \{\alpha(7\,;16\,;13) \mid \alpha \in \mathbb{R}\}}$

b)

$$\begin{cases} x + y + z = 0 \\ -2x + 5y + 2z = 1 \\ 8x + y + 4z = -1 \end{cases} \iff \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{array}\right) \xrightarrow[L_3 = L_3 - 8L_1]{L_2 = 2L_1 + L_2}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 1 \\ 0 & -7 & -4 & -1 \end{array}\right) \xrightarrow[L_3 = L_3 + L_2]{L_1 = 4L_1 - L_2} \left(\begin{array}{ccc|c} 4 & -3 & 0 & -1 \\ 0 & 7 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[L_2 = \frac{1}{4} L_2]{L_1 = \frac{1}{4} L_1}$$

$$\left(\begin{array}{ccc|c} 1 & -3/4 & 0 & -1/4 \\ 0 & 7/4 & 1 & 1/4 \\ 0 & 0 & 0 & 0 \end{array}\right) \iff \begin{cases} x - \frac{3}{4} y = -\frac{1}{4} \\ \frac{7}{4} y + z = \frac{1}{4} \end{cases}$$

posons $\frac{y}{4} = \alpha \Rightarrow \begin{cases} x - 3\alpha = -\frac{1}{4} \\ 7\alpha + z = \frac{1}{4} \end{cases} \Rightarrow \begin{cases} x = -\frac{1}{4} + 3\alpha \\ z = \frac{1}{4} - 7\alpha \end{cases}$

$\Rightarrow y = 4\alpha$

$$\Rightarrow \boxed{S = \left\{ \left(-\frac{1}{4} ; 0 ; \frac{1}{4}\right) + \alpha (3 ; 4 ; -7) \;\middle|\; \alpha \in \mathbb{R} \right\}}$$

c)

$$\begin{cases} x + y + 2z = 4 \\ 3x - y + z = 3 \\ -3x + 5y + 4z = 5 \end{cases} \iff \left(\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 3 & -1 & 1 & 3 \\ -3 & 5 & 4 & 5 \end{array}\right) \xrightarrow{L_2 = L_2 + L_3}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 4 & 5 & 8 \\ -3 & 5 & 4 & 5 \end{array}\right) \xrightarrow{L_3 = 3L_1 + L_3} \begin{pmatrix} 1 & 1 & 2 & 4 \\ 0 & 4 & 5 & 8 \\ 0 & 8 & 10 & 17 \end{pmatrix} \xrightarrow{L_2 \leftrightarrow L_3}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 8 & 10 & 17 \\ 0 & 4 & 5 & 8 \end{array}\right) \xrightarrow{L_2 = L_2 - 2L_3} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 4 & 5 & 8 \end{array}\right)$$

$\Rightarrow \boxed{S = \emptyset}$

d) $\begin{cases} 2x + y + 3z = 3 \\ 3x - y + 4z = 2 \\ 4x + y - z = 5 \\ x + y + z = 4 \end{cases} \Leftrightarrow \left(\begin{array}{ccc|c} 2 & 1 & 3 & 3 \\ 3 & -1 & 4 & 2 \\ 4 & 1 & -1 & 5 \\ 1 & 1 & 1 & 4 \end{array}\right) \xrightarrow[\substack{L_2 + L_3 \\ L_2 + L_4}]{L_1 + L_2}$

$\left(\begin{array}{ccc|c} 2 & 1 & 3 & 3 \\ 5 & 0 & 7 & 5 \\ 7 & 0 & 3 & 7 \\ 4 & 0 & 5 & 6 \end{array}\right) \xrightarrow[4L_2 - 5L_3]{7L_2 - 5L_3} \left(\begin{array}{ccc|c} 2 & 1 & 3 & 3 \\ 5 & 0 & 7 & 5 \\ 0 & 0 & 34 & 0 \\ 0 & 0 & 3 & -10 \end{array}\right) \xrightarrow[L_4 - 3L_3]{\frac{1}{34} L_3}$

$\left(\begin{array}{ccc|c} 2 & 1 & 3 & 3 \\ 5 & 0 & 7 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -10 \end{array}\right) \Rightarrow \boxed{S = \emptyset}$

e) $\begin{cases} x - 3y + z - t = 0 \\ 2x + y - z + 2t = 0 \end{cases} \Leftrightarrow \left(\begin{array}{cccc|c} 1 & -3 & 1 & -1 & 0 \\ 2 & 1 & -1 & 2 & 0 \end{array}\right) \xrightarrow[2L_1 + L_2]{L_1 + L_2}$

$\left(\begin{array}{cccc|c} 3 & -2 & 0 & 1 & 0 \\ 4 & -5 & 1 & 0 & 0 \end{array}\right) \Leftrightarrow \begin{cases} 3x - 2y + t = 0 \\ 4x - 5y + z = 0 \end{cases} \Leftrightarrow \begin{cases} 3x = 2y - t \\ 4x = 5y - z \end{cases}$

posons $\begin{cases} x = \alpha \\ y = \beta \end{cases} \Rightarrow 3\alpha = 2\beta - t \Rightarrow t = -3\alpha + 2\beta$

$\Rightarrow z = -4\alpha + 5\beta$

$\Rightarrow \boxed{S = \left\{ \alpha(1\,;0\,;-4\,;-3) + \beta(0\,;1\,;5\,;2) \,\middle|\, \alpha,\beta \in \mathbb{R} \right\}}$

f) $\begin{cases} x + 5y + 4z - 13t = 3 \\ 3x - y + 2z + 5t = 2 \\ 2x + 2y + 3z - 4t = 1 \end{cases} \Leftrightarrow \left(\begin{array}{cccc|c} 1 & 5 & 4 & -13 & 3 \\ 3 & -1 & 2 & 5 & 2 \\ 2 & 2 & 3 & -4 & 1 \end{array}\right) \xrightarrow[2L_1 - L_3]{3L_1 - L_2}$

$$\left(\begin{array}{cccc|c} 1 & 5 & 4 & -13 & 3 \\ 0 & 16 & 10 & -44 & 7 \\ 0 & 8 & 5 & -22 & 5 \end{array}\right) \xrightarrow{L_2 - 2L_3} \left(\begin{array}{cccc|c} 1 & 5 & 4 & -13 & 3 \\ 0 & 16 & 10 & -44 & 7 \\ 0 & 0 & 0 & 0 & -3 \end{array}\right)$$

$\rightarrow \boxed{S = \emptyset}$

1.1.13 Calculer l'inverse de chacune des matrices ci-dessous :

a) $\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}$ b) $\begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} \\ 1 & 1 & 0 \end{pmatrix}$ c) $\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 3 & 0 & 4 \\ 5 & 0 & 6 & 0 \\ 0 & 7 & 0 & 8 \end{pmatrix}$

a) $A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}$

* Calcul des cofacteurs :

- cofacteurs ligne 1 : $A_{11} = + \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$; $A_{12} = - \begin{vmatrix} 0 & -1 \\ 1 & 1 \end{vmatrix} = -1$

$A_{13} = + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$

- cofacteurs ligne 2 : $A_{21} = - \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = 1$; $A_{22} = + \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$

$A_{23} = - \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = -(0+1) = -1$

- cofacteurs ligne 3 : $A_{31} = + \begin{vmatrix} -1 & 0 \\ 1 & -1 \end{vmatrix} = 1$; $A_{32} = - \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = +1$

$$A_{33} = + \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$$

$$* \quad \mathrm{Det}\, A = \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} - 0 \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} + 1 \begin{vmatrix} -1 & 0 \\ 1 & -1 \end{vmatrix}$$

$$= 1 + 1 = 2$$

$$\Rightarrow \quad A^{-1} = \frac{1}{\det A} \, {}^t\!\begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \Rightarrow \boxed{A^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & 1 \end{pmatrix}}$$

ou : <u>détermination de l'inverse d'une matrice par la méthode du pivot.</u>

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{L_3 = L_3 - L_1} \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right) \xrightarrow{L_1 = L_1 + L_2}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right) \xrightarrow{L_3 = L_3 + L_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 2 & 0 & -1 & 1 & 1 \end{array}\right) \xrightarrow{L_3 = \frac{1}{2} L_3}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right) \xrightarrow{L_3 = L_3 - L_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}\right) \xrightarrow{L_1 = L_1 + L_3}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}\right) \xrightarrow{L_2 = L_2 + L_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\begin{matrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{matrix}}_{C = A^{-1}} \\ 0 & 0 & 1 & \end{array}\right)$$

$$\Rightarrow \quad C = A^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & 1 \end{pmatrix}$$

b)

$$A = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} \\ 1 & 1 & 0 \end{pmatrix}$$

$$\det A = \begin{vmatrix} -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} \\ 1 & 1 & 0 \end{vmatrix} = -\frac{1}{2} \begin{vmatrix} \frac{1}{2} & -\frac{1}{2} \\ 1 & 0 \end{vmatrix} - \frac{3}{2} \begin{vmatrix} -\frac{1}{2} & -\frac{1}{2} \\ 1 & 0 \end{vmatrix} + 1 \begin{vmatrix} -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix}$$

$$= -\frac{1}{2}\left(+\frac{1}{2}\right) - \frac{3}{2}\left(\frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{4}\right) = -\frac{1}{4} - \frac{3}{4} + \frac{2}{4} = \frac{-1-3+2}{4}$$

$$= -\frac{2}{4} = -\frac{1}{2}$$

$$\Rightarrow A^{-1} = \frac{1}{-\frac{1}{2}} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -1 \\ 1 & 0 & \frac{1}{2} \end{pmatrix} = -2 \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -1 \\ 1 & 0 & \frac{1}{2} \end{pmatrix}$$

$$\Rightarrow \boxed{A^{-1} = \begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & 2 \\ -2 & 0 & -1 \end{pmatrix}}$$

c)

$$A = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 3 & 0 & 4 \\ 5 & 0 & 6 & 0 \\ 0 & 7 & 0 & 8 \end{pmatrix}$$

$$A^{-1} = \frac{1}{144 - 168 - 240 + 280} \begin{pmatrix} -24 & 0 & 8 & 0 \\ 0 & -32 & 0 & 16 \\ 20 & 0 & -4 & 0 \\ 0 & 28 & 0 & -12 \end{pmatrix}$$

$$\boxed{A^{-1} = \frac{1}{4} \begin{pmatrix} -6 & 0 & 2 & 0 \\ 0 & -8 & 0 & 4 \\ 5 & 0 & -1 & 0 \\ 0 & 7 & 0 & -3 \end{pmatrix}}$$

1.1.14 Utiliser la méthode de Gauss-Jordan pour montrer que si $abc \neq 0$, alors

$$\begin{pmatrix} 0 & 0 & a \\ 0 & b & 0 \\ c & 0 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 0 & \frac{1}{c} \\ 0 & \frac{1}{b} & 0 \\ \frac{1}{a} & 0 & 0 \end{pmatrix}$$

On a :

$$\left(\begin{array}{ccc|ccc} 0 & 0 & a & 1 & 0 & 0 \\ 0 & b & 0 & 0 & 1 & 0 \\ c & 0 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{L_1 \leftrightarrow L_3} \left(\begin{array}{ccc|ccc} c & 0 & 0 & 0 & 0 & 1 \\ 0 & b & 0 & 0 & 1 & 0 \\ 0 & 0 & a & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[\substack{L_2 = \frac{1}{b} L_2 \\ L_3 = \frac{1}{a} L_3}]{L_1 = \frac{1}{c} L_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1/c \\ 0 & 1 & 0 & 0 & 1/b & 0 \\ 0 & 0 & 1 & 1/a & 0 & 0 \end{array}\right)$$

1.1.15 Résoudre les systèmes suivants à l'aide d'un calcul d'inverse de matrice :

a) $\begin{cases} 2x + 2y + 3z = 5 \\ x + z = -2 \\ x + y + z = 3 \end{cases}$ b) $\begin{cases} 3x + 5y + 4z = 3 \\ 4x + 5y + 4z = 0 \\ -x - y - z = 2 \end{cases}$ c) $\begin{cases} x + 2y + 3z = 1 \\ 2x - y - 2z = 5 \\ 3x + 5y + z = 10 \end{cases}$

a) $A = \begin{pmatrix} 2 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ $\det(A) = -2 + 1 + 2 = 1$

$\Rightarrow A^{-1} = \begin{pmatrix} -1 & 1 & 2 \\ 0 & -1 & 1 \\ 1 & 0 & -2 \end{pmatrix}$

donc $\begin{pmatrix} -1 & 1 & 2 \\ 0 & -1 & 1 \\ 1 & 0 & -2 \end{pmatrix} \begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ -1 \end{pmatrix} \Rightarrow \boxed{S = \{(-1 ; 5 ; -1)\}}$

* Autre méthode pour calculer A^{-1}

$$\left(\begin{array}{ccc|ccc} 2 & 2 & 3 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[L_2 = L_3 - L_2 \quad L_3 = L_2 - 2L_3]{L_1 = L_1 - L_3} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 & -2 \end{array}\right)$$

$$\xrightarrow{L_1 = L_1 - L_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 1 & -2 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 & -2 \end{array}\right) \xrightarrow{L_1 = L_1 - 2L_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & +2 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 & -2 \end{array}\right)$$

$$\Rightarrow C = A^{-1} = \begin{pmatrix} -1 & 1 & 2 \\ 0 & -1 & 1 \\ 1 & 0 & -2 \end{pmatrix}$$

donc $\begin{pmatrix} -1 & 1 & 2 \\ 0 & -1 & 1 \\ 1 & 0 & -2 \end{pmatrix} \begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ -1 \end{pmatrix} \Rightarrow \boxed{S = \{(-1\,;5\,;-1)\}}$

b) $B = \begin{pmatrix} 3 & 5 & 4 \\ 4 & 5 & 4 \\ -1 & -1 & -1 \end{pmatrix} \Rightarrow \det(B) = -3 + 4 = 1$

$$\Rightarrow B^{-1} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 4 \\ 1 & -2 & -5 \end{pmatrix}$$

$\Rightarrow \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 4 \\ 1 & -2 & -5 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 8 \\ 7 \end{pmatrix} \Rightarrow \boxed{S = \{(-3\,;8\,;7)\}}$

c)

$$C = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & -2 \\ 3 & 5 & 1 \end{pmatrix} \Rightarrow \det(C) = 9 + 26 - 3 = 32$$

$$\Rightarrow C^{-1} = \frac{1}{32} \begin{pmatrix} 9 & 13 & -1 \\ -8 & -8 & 8 \\ 13 & 1 & -5 \end{pmatrix}$$

$$\Rightarrow \frac{1}{32} \begin{pmatrix} 9 & 13 & -1 \\ -8 & -8 & 8 \\ 13 & 1 & -5 \end{pmatrix} \begin{pmatrix} 1 \\ 5 \\ 10 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \Rightarrow \boxed{S = \{ (2\,;\,1\,;\,-1) \}}$$