1.5 Nombres complexes sous forme exponentielle

1.5.1 Démontrer les formules suivantes. Pour tout $\theta \in \mathbb{R}$, on a :

a) $\overline{e^{i\theta}} = e^{-i\theta}$, $|e^{i\theta}| = 1$ et $(e^{i\theta})^{-1} = e^{-i\theta}$

b) $\cos(\theta) = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$ et $\sin(\theta) = \dfrac{e^{i\theta} - e^{-i\theta}}{2\,i}$

Rappel :

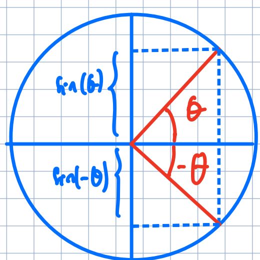
$$r\left(\cos(\theta) + i\sin(\theta)\right) = re^{i\theta}$$

$$\Rightarrow \cos(\theta) + i\sin(\theta) = e^{i\theta}$$

a) $\overline{e^{i\theta}} = e^{-i\theta}$

On a : * $e^{i\theta} = \cos(\theta) + i\sin(\theta)$

$$\Rightarrow \overline{e^{i\theta}} = \cos(\theta) - i\sin(\theta) = \cos(-\theta) + i\sin(-\theta) = e^{-i\theta}$$ ✓



** $|e^{i\theta}| = \sqrt{\cos^2(\theta) + \sin^2(\theta)} = 1$ ✓

*** $(e^{i\theta})^{-1} = e^{-i\theta} = \dfrac{1}{e^{i\theta}} = \dfrac{1}{\cos(\theta) + i\sin(\theta)}$

$$= \frac{1}{\cos(\theta) + i\sin(\theta)} \cdot \frac{\cos(\theta) - i\sin(\theta)}{\cos(\theta) - i\sin(\theta)} = \frac{\cos(\theta) - i\sin(\theta)}{\underbrace{\cos^2(\theta) + \sin^2(\theta)}_{=1}}$$

$$= \cos(\theta) - i\sin(\theta) = \cos(-\theta) + i\sin(-\theta) = e^{-i\theta}$$ ✓

b) * $\cos(\theta) = \dfrac{e^{i\theta} + e^{-i\theta}}{2} = \dfrac{\cos(\theta) + i\cancel{\sin(\theta)} + \cos(\theta) - i\cancel{\sin(\theta)}}{2}$

$= \dfrac{2\cos(\theta)}{2} = \cos(\theta)$ ✓

** $\sin(\theta) = \dfrac{e^{i\theta} - e^{-i\theta}}{2i} = \dfrac{\cos(\theta) + i\sin(\theta) - \left(\cos(\theta) - i\sin(\theta)\right)}{2i}$

$= \dfrac{\cancel{\cos(\theta)} + i\sin(\theta) - \cancel{\cos(\theta)} + i\sin(\theta)}{2i} = \dfrac{2i\sin(\theta)}{2i} = \sin(\theta)$ ✓