1.3.3 Résoudre dans $\mathbb{C}$ les équations ci-dessous :

a) $z^4 - (6+3i)z^3 + (8+12i)z^2 = 0$

b) $z^3 + 2z^2 + (-4+4i)z + 16 + 16i = 0$, sachant que -4 est un zéro

c) $z^3 - 4z^2 + (8+i)z - 7 + i = 0$, sachant que $1-i$ est un zéro

a) $z^4 - (6+3i)z^3 + (8+12i)z^2 = 0$

$\Rightarrow z^2\left(z^2 - (6+3i)z + (8+12i)\right) = 0$

$\Rightarrow z = 0$ ou $z^2 - (6+3i)z + (8+12i) = 0$

$\Delta = (6+3i)^2 - 4(8+12i) = 36 + 36i - 9 - 32 - 48i$

$\Delta = -5 - 12i$

Déterminer $w = a + bi$ tq $w^2 = (a+bi)^2 = -5 - 12i = \Delta$

$\Rightarrow a^2 + 2abi - b^2 = -5 - 12i$

$\Rightarrow a^2 - b^2 + 2abi = -5 - 12i$

$$\begin{cases} a^2 - b^2 = -5 \\ 2ab = -12 \\ a^2 + b^2 = \sqrt{(-5)^2 + (-12)^2} = 13 \end{cases} \Leftrightarrow \begin{cases} a = \pm 2 \\ b = \pm 3 \\ ab = -6 \end{cases}$$

donc $w_1 = 2 - 3i$ ou $w_2 = -2 + 3i$

$\Rightarrow$ 2 solutions : $z_1 = \dfrac{6 + 3i + 2 - 3i}{2} = 4$

$z_2 = \dfrac{6 + 3i - 2 + 3i}{2} = \dfrac{4 + 6i}{2} = 2 + 3i$

Alors $\boxed{S = \{0 ; 4 ; 2 + 3i\}}$

b) $z^3 + 2z^2 + (-4+4i)z + 16 + 16i = 0$

Horner :

	1	2	$-4+4i$	$16+16i$
-4		-4	8	$-16-16i$
	1	-2	$4+4i$	0

donc $p(z) = (z+4)(z^2 - 2z + 4 + 4i) = 0$

$\Rightarrow$ $z = -4$ ou $z^2 - 2z + 4 + 4i = 0$

$\Delta = (-2)^2 - 4\cdot 1 \cdot (4+4i) = 4 - 4(4+4i)$

$\Delta = 4 - 16 - 16i = -12 - 16i$

On détermine $w = a + bi$ tq $w^2 = (a+bi)^2 = -12-16i = \Delta$

$\Rightarrow$ $a^2 - b^2 + 2abi = -12 - 16i$

$$\Rightarrow \begin{cases} a^2 - b^2 = -12 \\ 2ab = -16 \\ a^2 + b^2 = \sqrt{(-12)^2 + (-16)^2} = 20 \end{cases}$$

$$\Rightarrow \begin{cases} a = \pm 2 \\ b = \pm 4 \\ ab = -8 \end{cases}$$

donc $w_1 = 2 - 4i$ ou $w_2 = -2 + 4i$

$$\Rightarrow z_1 = \frac{2 + 2 - 4i}{2} = \frac{4-4i}{2} = 2 - 2i$$

$$z_2 = \frac{2 - 2 + 4i}{2} = \frac{4i}{2} = 2i$$

Alors : $\boxed{S = \{-4\ ;\ 2i\ ;\ 2-2i\}}$

c) $\underbrace{z^3 - 4z^2 + (8+i)z - 7 + i}_{= P(z)} = 0$ sachant que $1-i$ est un zéro

Horner :

$(1-i)(-3-i) = -3 - i + 3i + i^2 = -4+2i$

$(1-i)(4+3i) = 4 + 3i - 4i - 3i^2 = 7-i$

$$\begin{array}{c|ccc|c} & 1 & -4 & 8+i & -7+i \\ 1-i & & 1-i & -4+2i & 7-i \\ \hline & 1 & -3-i & 4+3i & 0 \end{array}$$

$\Rightarrow P(z) = \left(z-(1-i)\right)\left(z^2 + (-3-i)z + 4+3i\right) = 0$

$= \left(z-(1-i)\right)\left(z^2 - (3+i)z + 4+3i\right) = 0$

$\Rightarrow z = 1-i$ ou $z^2 - (3+i)z + 4 + 3i = 0$

$\Delta = (3+i)^2 - 4\cdot 1\cdot(4+3i) = 9 + 6i - 1 - 4(4+3i)$

$\Delta = 8 + 6i - 16 - 12i = -8 - 6i$

On détermine $w = (a+bi)$ tq $w^2 = (a+bi)^2 = -8-6i = \Delta$

$\Rightarrow a^2 - b^2 + 2abi = -8 - 6i$

$\Rightarrow \begin{cases} a^2 - b^2 = -8 \\ 2ab = -6 \\ a^2 + b^2 = \sqrt{(-8)^2 + (-6)^2} = 10 \end{cases} \Rightarrow \begin{cases} a = \pm 1 \\ b = \pm 3 \\ ab = -3 \end{cases}$

donc $w_1 = 1 - 3i$ ou $w_2 = -1 + 3i$

$\Rightarrow z_1 = \dfrac{3+i+1-3i}{2} = \dfrac{4-2i}{2} = 2-i$

$z_2 = \dfrac{3+i-1+3i}{2} = \dfrac{2+4i}{2} = 1+2i$

Donc $\boxed{S = \{1-i \,;\, 2-i \,;\, 1+2i\}}$