Exercice 1.3.2

a) $z^4 - 1$

* Dans $\mathbb{R}[z]$: $p = z^4 - 1 = (z^2-1)(z^2+1)$

$\Rightarrow p = (z-1)(z+1)(z^2+1)$

* Dans $\mathbb{C}[z]$:

$p = z^4 - 1 = (z-1)(z+1)(z^2+1)$

$\Rightarrow$ on cherche 2 racines de z^2+1

$z^2+1 = 0$

$\Delta = 0^2 - 4\cdot 1\cdot 1 = -4$

$\Rightarrow z_1 = \dfrac{-0 - i\sqrt{4}}{2} = -\dfrac{i\cdot 2}{2} = -i$

$\Rightarrow z_2 = \dfrac{-0 + i\sqrt{4}}{2} = \dfrac{i\cdot 2}{2} = i$

Donc $z^2+1 = (z-i)(z+i)$

$\Rightarrow p = (z-1)(z+1)(z-i)(z+i)$

b) $z^4 + 1$

$p = z^4 + 1$

* Dans $\mathbb{C}[z]$

(comme $z^2 + 1$)

$p = (z^2-i)(z^2+i)$

Calculer les racines carrées de $-i$ et i

$* \quad i = 0 + 1 \cdot i$

$\Rightarrow a = 0$

$b = 1$

$|i| = 1$

$$\Rightarrow \begin{cases} \cos(\theta) = \dfrac{a}{r} = \dfrac{0}{1} = 0 \\ \sin(\theta) = \dfrac{b}{r} = \dfrac{1}{1} = 1 \end{cases} \Rightarrow \theta = \frac{\pi}{2}$$

$$\Rightarrow i = \left[1 ; \frac{\pi}{2}\right]$$

$w = [r, \theta]$

$\Rightarrow w^2 = [r^2 ; 2\theta]$

$$w^2 = i \iff \begin{cases} r^2 = 1 \\ 2\theta = \dfrac{\pi}{2} + 2k\pi \end{cases} \iff \begin{cases} r^2 = 1 \\ \theta = \dfrac{\pi}{4} + k\pi \end{cases}$$

$\Rightarrow$ Les racines carrées de i sont : $\left[1 ; \frac{\pi}{4}\right]$ et $\underbrace{\left[1 ; \frac{\pi}{4} + \pi\right]}_{= \left[1 ; \frac{5\pi}{4}\right]}$

$$z_1 = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$$

$$z_2 = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i$$

$* -i = [1 ; \frac{3\pi}{2}]$

$(-i \Rightarrow a = 0 , b = -1 \Rightarrow r = 1$

$$\begin{cases} \cos(\theta) = \frac{a}{r} = \frac{0}{1} = 0 \\ \sin(\theta) = \frac{b}{r} = \frac{-1}{1} = -1 \end{cases}$$

$\sin(\frac{\pi}{2}) = 1 \Rightarrow \sin(\underbrace{\pi + \frac{\pi}{2}}) = -\sin(\frac{\pi}{2}) = -1$

$$\left.\begin{array}{l} \sin(\frac{3\pi}{2}) = -1 \\ \text{et } \cos(\frac{3\pi}{2}) = 0 \end{array}\right\} \Rightarrow \theta = \frac{3\pi}{2} \Bigg)$$

$\Rightarrow -i = [1 ; \frac{3\pi}{2}]$

$w^2 = -i \iff \begin{cases} r^2 = 1 \\ 2\theta = \frac{3\pi}{2} + 2K\pi \end{cases}$

$w = [r ; \theta] \Rightarrow w^2 = [r^2 ; 2\theta]$

$\Rightarrow \begin{cases} r = 1 \\ \theta = \frac{3\pi}{4} + K\pi \end{cases}$

Les racines carrées de $-i$: $[1 ; \frac{3\pi}{4}]$ et $[1 ;$

$\Rightarrow z_3 = r(\cos(\theta) + i\sin(\theta)) = \underbrace{\cos(\frac{3\pi}{4})} + i\sin(\frac{3\pi}{4})$

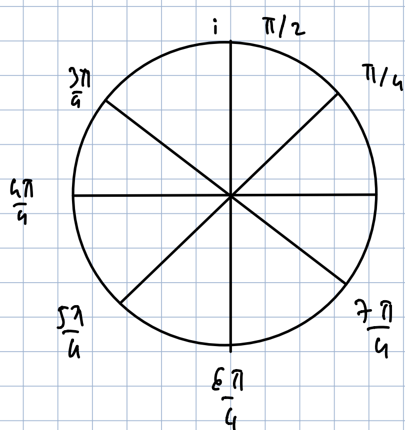
$= -\cos(\pi - \frac{3\pi}{4}) + i\sin(\pi - \frac{3\pi}{4})$

$$= -\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} = \overline{z_2}$$

$$\Rightarrow z_4 = r\left(\cos(\theta) + i\sin(\theta)\right) = 1\cdot\left(\cos\left(\frac{3\pi}{4}+\pi\right) + i\sin\left(\frac{3\pi}{4}+\pi\right)\right)$$

$$= \cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right) = \frac{\sqrt{2}}{2} + i\left(-\frac{\sqrt{2}}{2}\right)$$

$$z_4 = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i = \overline{z_1}$$



Donc

$$\boxed{z^4 + 1 = \left(z - \underbrace{\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)}_{z_1}\right)\left(z - \underbrace{\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)}_{\overline{z_1}}\right)\left(z - \underbrace{\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)}_{z_2}\right)\left(z - \underbrace{\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)}_{\overline{z_2}}\right)}$$

$$\boxed{z^4 + 1 = \left(z^2 - \sqrt{2}z + 1\right)\left(z^2 + \sqrt{2}z + 1\right)}$$

c) $z^3 + 1$

$\Rightarrow \quad P = z^3 + 1$

$P(-1) = (-1)^3 + 1 = 0 \quad \Rightarrow \quad (z+1) / P$

par Horner :

	1	0	0	1
-1		-1	1	-1
	1	-1	1	0

Donc $P = (z+1)\underbrace{(z^2 - z + 1)}_{\Delta < 0} \in \mathbb{R}[z]$

$\Rightarrow$ il faut factoriser $z^2 - z + 1$ dans $\mathbb{C}[z]$

$\Leftrightarrow$ résoudre l'équation complexe : $z^2 - z + 1 = 0$

$\Delta = -3$

$\Rightarrow$ les racines carrées de -3 sont $\pm\sqrt{3}\,i$

$\Rightarrow$ les zéros sont $\frac{1 - \sqrt{3}i}{2}$ et $\frac{1 + \sqrt{3}i}{2}$

Donc $\boxed{P = (z+1)\left(z - \frac{1-\sqrt{3}i}{2}\right)\left(z - \frac{1+\sqrt{3}i}{2}\right) \in \mathbb{C}[z]}$

d) $z^6 - 1$ Rappel : $a^3 - b^3 = (a-b)(a^2+ab+b^2)$

$$P = z^6 - 1 = (z^2)^3 - 1 = (z^2-1)(z^4+z^2+1)$$

$$\Rightarrow P = (z-1)(z+1)(z^4+z^2+1)$$

* $P = (z^3-1)(z^3+1)$

où $z^3 - 1 = (z-1)(z^2+z+1)$

et $z^3 + 1 = (z+1)(z^2-z+1)$

$$z^2 - z + 1 = \left(z - \frac{1-\sqrt{3}i}{2}\right)\left(z - \frac{1+\sqrt{3}i}{2}\right) \quad \text{(c.q.f. 1.3.2.c)}$$

et $z^2 + z + 1 = 0$ dans $\mathbb{C}[z]$

$\Delta = 1 - 4\cdot 1 = -3$

$\Rightarrow$ les racines carrées de -3 sont $\pm\sqrt{3}i$

$$\Rightarrow z^2 + z + 1 = \left(z - \frac{-1-\sqrt{3}i}{2}\right)\left(z - \frac{-1+\sqrt{3}i}{2}\right)$$

donc

$$z^6 - 1 = (z-1)(z+1)\underbrace{\left(z - \frac{1-\sqrt{3}i}{2}\right)\left(z - \frac{1+\sqrt{3}i}{2}\right)\left(z - \frac{-1-\sqrt{3}i}{2}\right)\left(z - \frac{-1+\sqrt{3}i}{2}\right)}_{z^4+z^2+1}$$

1) $z^6 + 9z^4 + 27z^2 + 27$

$P = z^6 + 9z^4 + 27z^2 + 27$

$P(3i) = -729 + 729 - 243 + 27 \neq 0$

$P(\sqrt{3}i) = -27 + 81 - 81 + 27 = 0$

$\Rightarrow z_1 = \sqrt{3}i$ et $z_2 = \overline{z_1} = -\sqrt{3}i$

$\Rightarrow (z - z_1)(z - z_2) = (z - \sqrt{3}i)(z + \sqrt{3}i) = z^2 + 3$

Division :

$$\begin{array}{rrrr|l}
z^6 & + 9z^4 & + 27z^2 & + 27 & z^2 + 3 \\
\cline{5-5}
z^6 & + 3z^4 & & & z^4 + 6z^2 + 9 \\
\cline{1-2}
 & 6z^4 & + 27z^2 & + 27 & \\
 & 6z^4 & + 18z^2 & & \\
\cline{2-4}
 & & 9z^2 & + 27 & \\
 & & 9z^2 & + 27 & \\
\cline{3-4}
 & & & 0 &
\end{array}$$

$\Rightarrow P = (z - \sqrt{3}i)(z + \sqrt{3}i)(\underbrace{z^4 + 6z^2 + 9})$

$P = (z^2 + 3) \qquad (z^2 + 3)^2 = (z^2 + 3)^3 \in \mathbb{R}[z]$

$P = (z - \sqrt{3}i)^3 (z + \sqrt{3}i)^3 \in \mathbb{C}[z]$