1.3 Propriétés algébriques des nombres complexes

1.3.1 Résoudre dans $\mathbb{C}$ les équations ci-dessous :

a) $z^2 = 25$

b) $z^2 = -4$

c) $2z^2 + 10z + 17 = 0$

d) $z^2 + 3z - 5 = 0$

e) $z^2 - 3(1+i)z + 6 + 7i = 0$

f) $(1+2i)z^2 - (7+4i)z + 5 - 5i = 0$

<u>Rappel :</u> si $z^2 = a$

- si $a > 0 \Rightarrow z = \pm\sqrt{a}$
- si $a < 0 \Rightarrow z = \pm\sqrt{|a|}\, i$

a) $z^2 = 25$

$\Rightarrow z = \pm 5 \Rightarrow \boxed{S = \{-5 ; 5\}}$

b) $z^2 = -4$

$\Rightarrow z = \pm 2i \Rightarrow \boxed{S = \{-2i ; +2i\}}$

c) $2z^2 + 10z + 17 = 0$

$\Rightarrow \Delta = 10^2 - 4 \cdot 2 \cdot 17 = 100 - 136 = -36 < 0$

$\Rightarrow z_1 = \dfrac{-10 - 6i}{4} = \dfrac{-5 - 3i}{2} = -\dfrac{5}{2} - \dfrac{3}{2} i$

$\Rightarrow z_2 = \dfrac{-10 + 6i}{4} = \dfrac{-5 + 3i}{2} = -\dfrac{5}{2} + \dfrac{3i}{2}$

$\Rightarrow \boxed{S = \left\{ -\dfrac{5}{2} \pm \dfrac{3i}{2} \right\}}$

d) $z^2 + 3z - 5 = 0$

$\Rightarrow \Delta = 9 - 4(-5) = 29 \Rightarrow z_{1,2} = \dfrac{-3 \pm \sqrt{29}}{2}$

$$\boxed{S = \left\{ \frac{-3 \pm \sqrt{29}}{2} \right\}}$$

e) $z^2 - 3(1+i)z + 6 + 7i = 0$

Rappel :

Soit a, b et $c \in \mathbb{C}$, on résout l'équation :

$$az^2 + bz + c = 0 \quad , \quad a \neq 0$$

$$= a\left(z^2 + \frac{b}{a}z + \frac{c}{a}\right)$$

$$= a\left(\left(z + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right)$$

Posons $D = \dfrac{b^2 - 4ac}{4a^2}$, il existe $d \in \mathbb{C}$ tq $d^2 = D$

$$az^2 + bz + c = 0$$

$$\Leftrightarrow a\left(z + \frac{b+d}{2a}\right)\left(z + \frac{b-d}{2a}\right) = 0$$

$\Rightarrow$ On déduit que toute équation complexe de degré 2 a deux solutions distinctes ou confondues.

ici, on a : $z^2 - 3(1+i)z + 6 + 7i = 0$

$$\Delta = (-3(1+i))^2 - 4 \cdot 1 (6 + 7i) = 9(1 + 2i + i^2) - 24 - 28i$$

$$\Delta = 18i - 24 - 28i = -24 - 10i$$

Déterminer $w = a + bi$ tq $w^2 = (a+bi)^2 = -24 - 10i = \Delta$

$\Rightarrow a^2 + 2abi + (bi)^2 = -24 - 10i$

$\Rightarrow a^2 - b^2 + 2abi = -24 - 10i$

par identification : $\begin{cases} a^2 - b^2 = -24 & \leftarrow \text{partie réelle} \\ 2ab = -10 & \leftarrow \text{partie imaginaire} \\ a^2 - b^2 = \sqrt{(-24)^2 + (-10)^2} & \leftarrow \text{module} \end{cases}$

$$\Rightarrow \begin{cases} a^2 - b^2 = -24 \\ ab = -5 \\ a^2 - b^2 = 26 \end{cases} \Rightarrow \begin{cases} a = \pm 1 \\ b = \pm 5 \\ ab = -5 \end{cases}$$

donc $a = 1$, $b = -5$ ou $a = -1$, $b = 5$

$\Rightarrow w = 1 - 5i$ ou $-1 + 5i$

$\Rightarrow$ on a 2 solutions :

$$z_1 = \frac{3(1+i) + 1 - 5i}{2} = \frac{4 - 2i}{2} = 2 - i$$

$$z_2 = \frac{3(1+i) - 1 + 5i}{2} = \frac{2 + 8i}{2} = 1 + 4i$$

$$\boxed{S = \{ 2 - i \ ; \ 1 + 4i \}}$$

1) $(1+2i)z^2 - (7+4i)z + 5-5i = 0$

$$\Delta = \left(-(7+4i)\right)^2 - 4(1+2i)(5-5i) = -27+36i$$

Déterminer $w = a+bi$ tq $w^2 = (a+bi)^2 = -27+36i = \Delta$

$\Rightarrow a^2 - b^2 + 2abi = -27+36i$

par identification : $\begin{cases} a^2 - b^2 = -27 \\ 2ab = 36 \\ a^2+b^2 = \sqrt{(-27)^2+36^2} = 45 \end{cases}$

$\Rightarrow \begin{cases} a = \pm 3 \\ b = \pm 6 \\ ab = 18 \end{cases}$ donc $w = 3+6i$ ou $w = -3-6i$

$\Rightarrow$ On a 2 solutions :

$$z_1 = \frac{7+4i+3+6i}{2(1+2i)} = \frac{10+10i}{2(1+2i)} = \frac{5+5i}{1+2i} \cdot \frac{1-2i}{1-2i} = \frac{15-5i}{5}$$

$\Rightarrow z_1 = 3-i$

et $$z_2 = \frac{7+4i-3-6i}{2(1+2i)} = \frac{4-2i}{2(1+2i)} = \frac{2-i}{1+2i} \cdot \frac{1-2i}{1-2i} = \frac{-5i}{5} = -i$$

$\Rightarrow \boxed{S = \{3-i\,;\,-i\}}$