1.2.8 Déterminer :

a) la forme trigonométrique des nombres complexes z tel que $z^4 = 1 + i$,

b) la forme algébrique des nombres complexes z tel que $z^4 = 24i - 7$.

a) la forme trigonométrique des nombres complexes z tel que

$$z^4 = 1 + i$$

$\Rightarrow \quad 1 + i = w \quad \Rightarrow \quad |w| = \sqrt{2}$

$a = 1 \quad , b = 1$

$$\Rightarrow \begin{cases} \cos(\theta) = \frac{a}{r} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \\ \sin(\theta) = \frac{b}{r} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \end{cases} \Rightarrow \theta = \frac{\pi}{4}$$

$\Rightarrow \quad w = 1 + i = [\sqrt{2} ; \frac{\pi}{4}]$

Soit $z = [r, \theta]$ tel que $z = [\sqrt{2} ; \frac{\pi}{4}]$

Donc $z^4 = [r^4 ; 4\theta] = [\sqrt{2} ; \frac{\pi}{4}]$

$$\Rightarrow \begin{cases} r^4 = \sqrt{2} \\ 4\theta = \frac{\pi}{4} + 2k\pi \end{cases} \Rightarrow \begin{cases} r = \sqrt[4]{\sqrt{2}} = \sqrt[8]{2} \\ \theta = \frac{\pi}{16} + k \cdot \frac{\pi}{2} \end{cases}, k \in \mathbb{Z}$$

$\Rightarrow$ 4 solutions :

$[\sqrt[8]{2} ; \frac{\pi}{16}]$; $[\sqrt[8]{2} ; \frac{9\pi}{16}]$; $[\sqrt[8]{2} ; \frac{17\pi}{16}]$

et $[\sqrt[8]{2} ; \frac{25\pi}{16}]$

b) La forme algébrique des nombres complexes z tel que $z^4 = 24i - 7$

On pose $w = -7 + 24i$

$\Rightarrow a = -7 \ , \ b = 24 \ , \ |w| = \sqrt{(-7)^2 + 24^2} = 25$

$$\Rightarrow \begin{cases} \cos(\theta) = \dfrac{a}{r} = \dfrac{-7}{25} \simeq -0{,}28 \\ \sin(\theta) = \dfrac{b}{r} = \dfrac{24}{25} \simeq 0{,}96 \end{cases} \Rightarrow \theta = 106{,}26^\circ$$

$$\Rightarrow w = [25 \ ; \ 106{,}26^\circ]$$

$z = a + bi = [r \ ; \ \theta] \Rightarrow z^4 = [r^4 \ ; \ 4\theta]$

$$z^4 = w \iff \begin{cases} r^4 = 25 \\ 4\theta = 106{,}26^\circ + k360^\circ \end{cases}$$

$$\Rightarrow \begin{cases} r = \sqrt{5} \\ \theta = \underbrace{26{,}57^\circ}_{\alpha} + k\,90^\circ \end{cases}$$

Donc $z_1 = \sqrt{5}\left(\cos(\alpha) + i\sin(\alpha)\right)$

$= \sqrt{5}\left(\cos(26{,}57^\circ) + i\sin(26{,}57^\circ)\right)$

$z_1 \simeq 2 + i$

$z_2 = \sqrt{5}\left(\cos(116{,}57^\circ) + i\sin(116{,}57^\circ)\right)$

$z_2 = -1 + 2i$

$z_3 = \sqrt{5}\left(\cos(206{,}57^\circ) + i\sin(206{,}57^\circ)\right)$

$z_3 = -2 - i$

$z_4 = 1 - 2i$