Exercice 1.2.10 :

Déterminer sous forme algébrique les racines carrées complexes des nombres ci-dessous :

a) 1

$\Rightarrow \quad z^2 = 1 \quad \Rightarrow \quad \sqrt{z^2} = \pm 1 \quad \Rightarrow \quad z = \pm 1$

$\Rightarrow \quad S = \{ -1 \,;\, 1 \}$

b) i

$\Rightarrow \quad z^2 = i \,; \quad w = 0 + 1 \cdot i \quad \Rightarrow \quad |w| = 1$

on pose $z = a + bi \quad \Rightarrow \quad |z|^2 = a^2 + b^2$

$\Rightarrow \quad (a + bi)^2 = 0 + 1i$

$\Rightarrow \quad a^2 - b^2 + 2abi = 0 + 1i$

$\Rightarrow \begin{cases} a^2 - b^2 = 0 & \text{partie réelle} \\ 2ab = 1 & \text{partie imaginaire} \\ a^2 + b^2 = 1 & \text{module} \end{cases}$

$\Rightarrow \begin{cases} a^2 - b^2 = 0 \\ a^2 + b^2 = 1 \\ 2ab = 1 \end{cases} \Rightarrow \begin{cases} 2a^2 = 1 \\ 2b^2 = 1 \\ 2ab = 1 \end{cases} \Rightarrow \begin{cases} a^2 = \frac{1}{2} \\ b^2 = \frac{1}{2} \\ ab = \frac{1}{2} \end{cases}$

$\Rightarrow \begin{cases} a = \pm \frac{1}{\sqrt{2}} \\ b = \pm \frac{1}{\sqrt{2}} \\ ab = \frac{1}{2} \end{cases}$ d'où $z_1 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i$

$z_2 = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i$

c) $-i$

$\Rightarrow z^2 = -i$, $w = -i = 0 - 1 \cdot i \Rightarrow |w| = 1$

On pose $z = a + bi \Rightarrow z^2 = a^2 - b^2 + 2abi$

d'où $a^2 - b^2 + 2abi = 0 - 1 \cdot i$

$$\Rightarrow \begin{cases} a^2 - b^2 = 0 \\ 2ab = -1 \\ a^2 + b^2 = 1 \end{cases} \Rightarrow \begin{cases} a^2 - b^2 = 0 \\ a^2 + b^2 = 1 \\ 2ab = -1 \end{cases} \Rightarrow \begin{cases} 2a^2 = 1 \\ -2b^2 = -1 \\ 2ab = -1 \end{cases}$$

$$\Rightarrow \begin{cases} a^2 = \frac{1}{2} \\ b^2 = \frac{1}{2} \\ 2ab = -1 \end{cases} \Rightarrow \begin{cases} a = \pm \frac{1}{\sqrt{2}} = \pm \frac{\sqrt{2}}{2} \\ b = \pm \frac{1}{\sqrt{2}} = \pm \frac{\sqrt{2}}{2} \\ ab = -\frac{1}{2} \end{cases}$$

d'où $z_1 = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i$ et $z_2 = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$

d) -9

$\Rightarrow z^2 = -9 \Rightarrow z = \pm 3i$

e) $3 + 4i$

$w = 3 + 4i \Rightarrow |w| = 5$

On pose $z = a + bi \Rightarrow z^2 = a^2 - b^2 + 2abi$

d'où $z^2 = w$

$$\begin{cases} a^2 - b^2 = 3 \\ 2ab = 4 \\ a^2 + b^2 = 5 \end{cases} \Rightarrow \begin{cases} a^2 - b^2 = 3 \\ a^2 + b^2 = 5 \\ ab = 2 \end{cases}$$

$$\begin{cases} 2a^2 = 8 \\ -2b^2 = -2 \\ ab = 2 \end{cases} \Rightarrow \begin{cases} a^2 = 4 \\ b^2 = 1 \\ ab = 2 \end{cases} \Rightarrow \begin{cases} a = \pm 2 \\ b = \pm 1 \\ ab = 2 \end{cases}$$

d'où $z_1 = 2 + i$ et $z_2 = -2 - i$

f) $-5 + 12i$

$w = -5 + 12i \Rightarrow |w| = 13$

On pose $z = a + bi \Rightarrow z^2 = a^2 - b^2 + 2abi$

$z^2 = w$

$$\Rightarrow \begin{cases} a^2 - b^2 = -5 \\ 2ab = 12 \\ a^2 + b^2 = 13 \end{cases} \Rightarrow \begin{cases} a^2 - b^2 = -5 \\ a^2 + b^2 = 13 \\ ab = 6 \end{cases} \Rightarrow \begin{cases} 2a^2 = 8 \\ -2b^2 = -18 \\ ab = 6 \end{cases}$$

$$\Rightarrow \begin{cases} a^2 = 4 \\ b^2 = 9 \\ ab = 6 \end{cases} \Rightarrow \begin{cases} a = \pm 2 \\ b = \pm 3 \\ ab = 6 \end{cases}$$

D'où les racines carrées sont : $z_1 = 2 + 3i$ et $z_2 = -2 - 3i$