1.3 Primitives et intégrales

1.3.1 Pour chacune des questions ci-dessous, montrer que $F(x)$ est une primitive de $f(x)$:

a) $F(x) = \dfrac{-2}{\sqrt{x^7}}$ et $f(x) = \dfrac{7}{\sqrt{x^9}}$;

b) $F(x) = \dfrac{2x^2-1}{2-x^2} + 7$ et $f(x) = \dfrac{6x}{(2-x^2)^2}$;

c) $F(x) = \dfrac{2x}{\sqrt{x+1}} - 11$ et $f(x) = \dfrac{x+2}{\sqrt{(x+1)^3}}$;

d) $F(x) = \sqrt{1+x} + \sqrt{1-x} + c$, avec $c \in \mathbb{R}$ et $f(x) = \dfrac{\sqrt{1-x} - \sqrt{1+x}}{2\sqrt{1-x^2}}$.

Rappel :

$F(x)$ est une primitive de $f(x)$ si $F'(x) = f(x)$

a) $F'(x) = \left(-2x^{-\frac{7}{2}}\right)' = -2 \cdot \dfrac{-7}{2} x^{-\frac{7}{2}-1} = 7x^{-\frac{9}{2}} = \boxed{\dfrac{7}{\sqrt{x^9}}}$

b) $F'(x) = \dfrac{4x(2-x^2) - (2x^2-1)(-2x)}{(2-x^2)^2} = \boxed{\dfrac{6x}{(2-x^2)^2}}$

c) $F'(x) = \dfrac{2\sqrt{x+1} - 2x \cdot \frac{1}{2\sqrt{x+1}}}{x+1} = \boxed{\dfrac{x+2}{\sqrt{(x+1)^3}}}$

d) $F'(x) = \dfrac{1}{2\sqrt{1+x}} - \dfrac{1}{2\sqrt{1-x}} = \boxed{\dfrac{\sqrt{1-x} - \sqrt{1+x}}{2\sqrt{1-x^2}}}$

1.3.2 Vérifier les égalités suivantes :

a) $\displaystyle\int \left(-\frac{1}{2x^2} + \frac{x}{2}\right) dx = \frac{1}{2x} + \frac{x^2}{4} + c$, avec $c \in \mathbb{R}$;

b) $\displaystyle\int \left(4x^2 - \frac{7}{3} - \frac{14}{3x^3}\right) dx = \frac{4x^5 - 7x^3 + 7}{3x^2} + c$, avec $c \in \mathbb{R}$;

c) $\displaystyle\int \frac{1}{(x+1)\sqrt{x^2-1}}\, dx = \sqrt{\frac{x-1}{x+1}} + c$, avec $c \in \mathbb{R}$;

d) $\displaystyle\int \frac{2x+7}{\sqrt[3]{(x^2+7x+2)^4}}\, dx = \frac{-3}{\sqrt[3]{x^2+7x+2}} + c$, avec $c \in \mathbb{R}$.

a) $$\left(\frac{1}{2x} + \frac{x^2}{4} + c\right)' = -\frac{1}{2x^2} + \frac{2x}{4} + 0 = \boxed{-\frac{1}{2x^2} + \frac{x}{2}}$$ ✓

b) $$\left(\frac{4x^3}{3} - \frac{7x}{3} + \frac{7}{3x^2} + c\right)' = 4x^2 - \frac{7}{3} - \frac{14x}{3x^4} + 0 = \boxed{4x^2 - \frac{7}{3} - \frac{14}{3x^3}}$$ ✓

c) $$\left(\sqrt{\frac{x-1}{x+1}} + c\right)' = \frac{\frac{x+1-(x-1)}{(x+1)^2}}{2\cdot\sqrt{\frac{x-1}{x+1}}} + 0 = \frac{1}{(x+1)^2}\cdot\sqrt{\frac{x+1}{x-1}}$$

$$= \sqrt{\frac{x+1}{(x+1)^4\,(x-1)}} = \frac{1}{\sqrt{\underbrace{(x+1)^3}_{(x+1)^2(x+1)}(x-1)}} = \boxed{\frac{1}{(x+1)\sqrt{x^2-1}}}$$

d) $$\left(-3\left(x^2+7x+2\right)^{-\frac{1}{3}} + c\right)' = -3\,\frac{(-1)}{3}\left(x^2+7x+2\right)^{-\frac{1}{3}-1}\cdot(2x+7) + 0$$

$$= \left(x^2+7x+2\right)^{-\frac{4}{3}}\cdot(2x+7) = \boxed{\frac{2x+7}{\sqrt[3]{(x^2+7x+2)^4}}}$$

1.3.3 Calculer :

a) $\int 3\,dx$ $= 3x + C$

b) $\int 5x\,dx$ $= \frac{5}{2}x^2 + C$

c) $\int (2x+1)\,dx$ $= x^2 + x + C$

d) $\int (5x-4)\,dx$ $= \frac{5}{2}x^2 - 4x + C$

e) $\int (2x^2-3x+2)\,dx$ $= \frac{2}{3}x^3 - \frac{3}{2}x^2 + 2x + C$

f) $\int 5x^3\,dx$ $= \frac{5}{4}x^4 + C$

g) $\int (-3x^4)\,dx$ $= -\frac{3}{5}x^5 + C$

h) $\int (3x^5+2x^4-1)\,dx$ $= \frac{1}{2}x^6 + \frac{2}{5}x^5 - x + C$

i) $\int (\cos(x)+\sin(x))\,dx$

j) $\int (1+\tan^2(x))\,dx$ $= \tan(x) + C$

i) $\displaystyle\int (\cos(x) + \sin(x))\,dx = \int \cos(x)\,dx + \int \sin(x)\,dx$

$= \sin(x) - \cos(x) + C$

1.3.4 Calculer :

a) $\displaystyle\int \frac{dx}{x^2} = \int x^{-2}\,dx = \frac{x^{-1}}{-1} + C = -\frac{1}{x} + C$

b) $\displaystyle\int \frac{2dx}{x^3} = 2\frac{(-1)}{2}x^{-2} + C = -\frac{1}{x^2} + C$

c) $\displaystyle\int \frac{-7dx}{x^5} = -7\frac{(-1)}{4}x^{-4} + C = \frac{7}{4x^4} + C$

d) $\displaystyle\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}dx = \frac{2}{3}x^{\frac{3}{2}} + C$

$= \frac{2}{3}\sqrt{x^3} + C$

e) $\displaystyle\int \sqrt[3]{x}\,dx = \int x^{\frac{1}{3}}\,dx = \frac{3}{4}x^{\frac{4}{3}} + C$

$= \frac{3}{4}\sqrt[3]{x^4} + C$

f) $\displaystyle\int \frac{dx}{\sqrt{x}}$

g) $\displaystyle\int \frac{dx}{\sqrt[3]{x^2}} = \int \frac{dx}{x^{2/3}} = 3x^{\frac{1}{3}} + C = 3\sqrt[3]{x} + C$

h) $\displaystyle\int \left(\frac{3}{x^4} - \sqrt[4]{x^3}\right)dx$

f) $\displaystyle\int \frac{dx}{\sqrt{x}} = \int \frac{dx}{x^{\frac{1}{2}}} = \int x^{-\frac{1}{2}}\,dx = \frac{1}{\frac{1}{2}}x^{\frac{1}{2}} + C = 2\sqrt{x} + C$

h) $\displaystyle\int \left(\frac{3}{x^4} - \sqrt[4]{x^3}\right)dx = 3\left(\frac{-1}{3}\right)x^{-3} - \frac{4}{7}x^{\frac{7}{4}} + C = -\frac{1}{x^3} - \frac{4}{7}\sqrt[4]{x^7} + C$

1.3.5 Calculer :

a) $\displaystyle\int \cos(3x)\,dx$

b) $\displaystyle\int \sin\left(2x - \frac{\pi}{3}\right)\,dx$

c) $\displaystyle\int (x+3)^3\,dx$

d) $\displaystyle\int (2x-1)^2\,dx$

e) $\displaystyle\int (7x-2)^5\,dx$

f) $\displaystyle\int (3x^2+x)^3(6x+1)\,dx$

g) $\displaystyle\int (4x^2+3)^4 x\,dx$

h) $\displaystyle\int \sin^2(x)\cos(x)\,dx$

i) $\displaystyle\int \frac{\tan^2(x)}{\cos^2(x)}\,dx$

j) $\displaystyle\int \sqrt{x+3}\,dx$

k) $\displaystyle\int \frac{dx}{\sqrt{3x+1}}$

l) $\displaystyle\int \frac{x+1}{\sqrt{x^2+2x}}\,dx$

a) $$\int \cos(3x)\,dx = \frac{1}{3}\int \underbrace{\cos(3x)}_{\cos(f(x))} \cdot \underbrace{3}_{f'(x)}\,dx = \boxed{\frac{1}{3}\sin(3x) + C}$$

b) $$\int \sin\left(2x - \frac{\pi}{3}\right)dx = -\frac{1}{2}\int \underbrace{-\sin\left(2x-\frac{\pi}{3}\right)}_{-\sin(f(x))} \cdot \underbrace{2}_{f'(x)}\,dx = \boxed{-\frac{1}{2}\cos\left(2x-\frac{\pi}{3}\right)+C}$$

c) $$\int (x+3)^3\,dx = \int \underbrace{(x+3)^3}_{f^3(x)} \cdot \underbrace{1}_{f'(x)}\,dx = \boxed{\frac{1}{4}(x+3)^4 + C}$$

d) $$\int (2x-1)^2\,dx = \frac{1}{2}\int \underbrace{(2x-1)^2}_{f^2(x)} \cdot \underbrace{2}_{f'(x)}\,dx = \frac{1}{2}\,\frac{(2x-1)^3}{3} + C$$

$$= \boxed{\frac{1}{6}(2x-1)^3 + C}$$

e) $\displaystyle\int (7x-2)^5\,dx = \frac{1}{7}\int \underbrace{(7x-2)^5}_{f^5(x)} \cdot \underbrace{7}_{f'(x)}\,dx = \frac{1}{7}\,\frac{(7x-2)^6}{6} + C$

$$= \boxed{\frac{1}{42}(7x-2)^6 + C}$$

f) $\displaystyle\int \underbrace{(3x^2+x)^3}_{f^3(x)}\;\underbrace{(6x+1)}_{f'(x)}\,dx = \boxed{\frac{1}{4}(3x^2+x)^4 + C}$

g) $\displaystyle\int (4x^2+3)^4 \cdot x\,dx = \frac{1}{8}\int \underbrace{(4x^2+3)^4}_{f^4(x)} \cdot \underbrace{8x}_{f'(x)}\,dx$

$$= \frac{1}{8}\,\frac{(4x^2+3)^5}{5} + C = \boxed{\frac{1}{40}(4x^2+3)^5 + C}$$

h) $\displaystyle\int \underbrace{\sin^2(x)}_{f^2(x)}\;\underbrace{\cos(x)}_{f'(x)}\,dx = \boxed{\frac{1}{3}\sin^3(x) + C}$

i) $\displaystyle\int \frac{\tan^2(x)}{\cos^2(x)}\,dx = \int \underbrace{\tan^2(x)}_{f^2(x)} \cdot \underbrace{\frac{1}{\cos^2(x)}}_{f'(x)}\,dx = \boxed{\frac{1}{3}\tan^3(x) + C}$

j) $\displaystyle\int \sqrt{x+3}\,dx = \int \underbrace{(x+3)^{\frac{1}{2}}}_{f^{\frac{1}{2}}(x)} \cdot \underbrace{1}_{f'(x)}\,dx = \frac{(x+3)^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$

$$= \frac{(x+3)^{\frac{3}{2}}}{\frac{3}{2}} + C = \boxed{\frac{2}{3}\sqrt{(x+3)^2} + C}$$

k) $\displaystyle\int \frac{dx}{\sqrt{3x+1}} = \frac{1}{3}\int \underbrace{(3x+1)^{-\frac{1}{2}}}_{f^{-\frac{1}{2}}(x)} \cdot \underbrace{3}_{f'(x)}\, dx = \frac{1}{3}\,\frac{(3x+1)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C$

$\displaystyle = \frac{1}{3}\,\frac{(3x+1)^{\frac{1}{2}}}{\frac{1}{2}} + C = \frac{1}{3}\cdot 2 \cdot \sqrt{3x+1} + C$

$\displaystyle = \boxed{\frac{2}{3}\sqrt{3x+1} + C}$

l) $\displaystyle\int \frac{x+1}{\sqrt{x^2+2x}}\, dx = \frac{1}{2}\int \underbrace{(x^2+2x)^{-\frac{1}{2}}}_{f^{-\frac{1}{2}}(x)} \cdot \underbrace{2(x+1)}_{f'(x)}\, dx$

$\displaystyle = \frac{1}{2}\,\frac{(x^2+2x)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C = \frac{1}{2}\,\frac{(x^2+2x)^{\frac{1}{2}}}{\frac{1}{2}} + C$

$\displaystyle = \frac{2}{2}(x^2+2x)^{\frac{1}{2}} + C = \boxed{\sqrt{x^2+2x} + C}$