1.3.15 Calculer les intégrales définies suivantes.

a) $\displaystyle\int_1^2 \frac{x}{x+6}\,dx$

b) $\displaystyle\int_0^4 \sqrt{x}(x+2)\,dx$

c) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^5(x)\,\cos(x)\,dx$

d) $\displaystyle\int_2^{\sqrt{20}} 3x\sqrt{x^2+5}\,dx$

e) $\displaystyle\int_0^3 \sqrt{9-x^2}\,dx$

f) $\displaystyle\int_2^3 \frac{5x-2}{x^2-x}\,dx$

a) $\displaystyle\int_1^2 \frac{x}{x+6}\,dx$ $\Rightarrow$ Division

$$\begin{array}{r|l} x & x+6 \\ \hline -\ \underline{x+6} & 1 \\ 0\ -6 & \end{array}$$

$\Rightarrow \dfrac{x}{x+6} = 1 - \dfrac{6}{x+6}$

$\Rightarrow \displaystyle\int_1^2 \left(1 - \frac{6}{x+6}\right) dx = \int_1^2 dx - 6\int_1^2 \frac{1}{x+6}\,dx$

$= x\Big|_1^2 - 6\ln(|x+6|)\Big|_1^2$

$= 2-1-6\left(\ln(8)-\ln(7)\right) = \boxed{1+6\ln\left(\frac{7}{8}\right)}$

b) $$\int_0^4 \sqrt{x}\,(x+2)\,dx = \int_0^4 \left(x\sqrt{x} + 2\sqrt{x}\right) dx = \int_0^4 \left(x \cdot x^{\frac{1}{2}} + 2x^{\frac{1}{2}}\right) dx$$

$$= \int_0^4 \left(x^{\frac{3}{2}} + 2x^{\frac{1}{2}}\right) dx = \int_0^4 x^{\frac{3}{2}}\,dx + 2\int_0^4 x^{1/2}\,dx$$

$$= \frac{x^{\frac{5}{2}}}{\frac{5}{2}}\Bigg|_0^4 + 2\,\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_0^4 = \frac{2}{5}\,x^{\frac{5}{2}}\Big|_0^4 + \frac{4}{3}\,x^{\frac{3}{2}}\Big|_0^4$$

$$= \frac{2}{5}\sqrt{x^5}\Big|_0^4 + \frac{4}{3}\sqrt{x^3}\Big|_0^4 = \frac{64}{5} + \frac{32}{3} - 0$$

$$= \boxed{\frac{352}{15}}$$

c) $$\int_0^{\pi/2} \underbrace{\sin^5(x)}_{u^5}\,\underbrace{\cos(x)}_{u'}\,dx = \frac{\sin^6(x)}{6}\Bigg|_0^{\pi/2} = \frac{1}{6} - 0 = \boxed{\frac{1}{6}}$$

d) $$\int_2^{\sqrt{20}} 3x\sqrt{x^2+5}\,dx = \frac{1}{2}\int_2^{\sqrt{20}} 3\,\underbrace{\sqrt{x^2+5}}_{u^{1/2}} \cdot \underbrace{2x}_{u'}\,dx$$

$$= \frac{3}{2}\,\frac{\left(x^2+5\right)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\Bigg|_2^{\sqrt{20}} = \frac{3}{2}\cdot\frac{2}{3}\left(x^2+5\right)^{\frac{3}{2}}\Big|_2^{\sqrt{20}}$$

$$= \sqrt{\left(x^2+5\right)^3}\,\Big|_2^{\sqrt{20}} = 125 - 27 = \boxed{98}$$

e) $\int_0^3 \sqrt{9-x^2}\,dx$

On pose $x = 3\sin(t) \Rightarrow dx = 3\cos(t)\,dt$

$x = 0 \Rightarrow \sin(t) = 0 \Rightarrow t = 0$

$x = 3 \Rightarrow 3\sin(t) = 3 \Rightarrow \sin(t) = 1 \Rightarrow t = \frac{\pi}{2}$

$$\Rightarrow \int_0^3 \sqrt{9-x^2}\,dx = \int_0^{\pi/2} \sqrt{9-9\sin^2(t)}\;3\cos(t)\,dt$$

$$= \int_0^{\pi/2} \sqrt{9(1-\sin^2(t))}\;3\cos(t)\,dt = \int_0^{\pi/2} 3\sqrt{\cos^2(t)}\;3\cos(t)\,dt$$

$$= 9\int_0^{\pi/2} \cos^2(t)\,dt = 9\underbrace{\int_0^{\pi/2} \cos(t)\cos(t)\,dt}_{\text{intégration par parties} \to \text{ex } 2.2.1\text{c}}$$

$$= 9\left(\frac{t}{2} + \underbrace{\frac{1}{2}\cos(t)\sin(t)}_{\frac{1}{2}\sin(2t) = \cos(t)\sin(t)}\right)\Bigg|_0^{\pi/2}$$

$$= 9\left(\frac{t}{2} + \frac{1}{4}\sin(2t)\right)\Bigg|_0^{\pi/2} = \frac{9t}{2}\Bigg|_0^{\pi/2} + \frac{9}{4}\sin(2t)\Bigg|_0^{\pi/2}$$

$$= \frac{9}{2}\cdot\frac{\pi}{2} - 0 + \frac{9}{4}\sin\left(2\cdot\frac{\pi}{2}\right) - \frac{9}{4}\sin(2\cdot 0)$$

$$= \frac{9\pi}{4} - 0 + 0 - 0 = \boxed{\frac{9\pi}{4}}$$

f) $\int_2^3 \frac{5x-2}{x^2-x}\,dx$

$\Rightarrow$ Décomposition en fractions simples :

$$\frac{5x-2}{x^2-x} = \frac{5x-2}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1} = \frac{A(x-1)+Bx}{x(x-1)}$$

$\Leftrightarrow \quad 5x-2 = A(x-1) + Bx \Leftrightarrow 5x-2 = Ax - A + Bx$

$\Leftrightarrow \quad 5x-2 = (A+B)x - A$

$$\Rightarrow \begin{cases} A+B = 5 \\ -A = -2 \end{cases} \Leftrightarrow \begin{cases} B = 3 \\ A = 2 \end{cases}$$

D'où $\frac{5x-2}{x^2-x} = \frac{2}{x} + \frac{3}{x-1}$

$$\Rightarrow \int_2^3 \frac{5x-2}{x^2-x}\,dx = \int_2^3 \left(\frac{2}{x} + \frac{3}{x-1}\right)dx = 2\int_2^3 \frac{1}{x}\,dx + 3\int_2^3 \frac{1}{x-1}\,dx$$

$$= 2\ln(|x|)\Big|_2^3 + 3\ln(|x-1|)\Big|_2^3$$

$$= 2\ln(3) - 2\ln(2) + 3\ln(2) - 3\ln(1)$$

$$= \ln(3)^2 - \ln(2)^2 + \ln(2)^3 - 0$$

$$= \ln(9) - \ln(4) + \ln(8) = \underbrace{\ln(9) + \ln(8)}_{\ln(72)} - \ln(4)$$

$$= \ln(72) - \ln(4) = \ln\left(\frac{72}{4}\right) = \boxed{\ln(18)}$$